

TAU FUNCTIONS, BIRKHOFF FACTORIZATIONS & DIFFERENCE EQUATIONS

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ABSTRACT. Q -systems and T -systems are integrable difference equations that recently have attracted much attention, and have wide applications in representation theory and statistical mechanics. We show that certain τ -functions, given as matrix elements of the action of the loop group of \mathbf{GL}_2 on two component Fermionic Fock space, give solutions of Q -systems.

An obvious generalization using the loop group of \mathbf{GL}_3 acting on three component Fermionic Fock space leads to a new system of 4 difference equations, with hopefully also interesting applications.

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1. INTRODUCTION

Many integrable differential equations can be transformed in simpler, bilinear, form, by introducing new dependent variables called τ -functions. In practice, these τ -functions are given as matrix elements of infinite dimensional groups or Lie algebras, etc.

For instance, the famous Korteweg-de Vries equation

$$(1.1) \quad \partial_t u + u_{xxx} + 6u\partial_x u = 0$$

is transformed by the substitution

$$u = 2\partial_x^2 \ln(\tau)$$

into Hirota bilinear form

$$(D_x D_t + D_x^4)\tau \cdot \tau = 0,$$

where D_u is the Hirota operator so that $D_u \sigma \cdot \tau = (\partial_u \sigma)\tau - \sigma \partial_u \tau$. See [6] for details and many more examples. In the case of the KdV equation the τ -function is a matrix element for the action of the loop group of \mathbf{GL}_2 on one component Fermionic Fock space, see for instance [4], [13], [8].

To produce the integrable equations from τ -functions one introduces an intermediate object, the Baker function. It satisfies linear equations, and the compatibility of these equations gives the integrable hierarchy.

For instance, for the KdV case the τ -function is a scalar function $\tau(t_1, t_3, t_5, \dots)$ of odd times ($t_1 = x, t_3 = t$), and the Baker function is also a scalar function Ψ of the times t_{2k+1} and of an extra variable z , the spectral parameter. It is defined by

$$\Psi(z; t_1, t_3, \dots) = \Gamma(z, t)\tau(t)/\tau,$$

where $\Gamma(z, t) = e^{\sum z^k t_k} e^{-\sum \frac{1}{kz^k} \partial_{t_k}}$, essentially the vertex operator for the free fermion vertex algebra, see e.g., [9]. The Baker function satisfies linear equations

$$(1.2) \quad \partial_{t_k} \Psi(z; t) = B_k(\partial_x) \Psi(z; t),$$

where $B_k(\partial_x)$ is a degree k differential operator in ∂_x . The compatibility of (1.2) for $k = 1$ and $k = 3$ turns out to give precisely the Korteweg-de Vries equation (1.1).

In this paper we are interested in integrable *difference* (as opposed to differential) equations. Still, we follow very much the setup sketched above for the KdV hierarchy.

In the first part we introduce a collection of τ -functions as matrix elements of the action of loop group elements for \mathbf{GL}_2 , depending on discrete variables¹ c_k , which play a similar role as the higher KdV times t_{2k+1} , $k > 1$. These τ -functions are of the form $\tau_k^{(\alpha)}(c_k)$, where k, α are discrete variables. In fact, these τ -functions turn out to be (see Theorem 2.1) Hankel determinants, well known since the 19th century in the theory of orthogonal polynomials, see e.g., [7].

¹These variables can be thought of as coordinates on the lower triangular subgroup of the loop group of \mathbf{GL}_2

We then define Baker functions. They turn out to be in this case 2×2 matrices depending on a spectral parameter z and on the discrete variables (and on the c_k):

$$\Psi_k^{(\alpha)}(z) = \frac{1}{\tau_k^{(\alpha)}} \begin{bmatrix} z^k & 0 \\ 0 & z^{-k} \end{bmatrix} \begin{bmatrix} S^+(z) & 0 \\ 0 & S^-(z) \end{bmatrix} \begin{bmatrix} \tau_k^{(\alpha)} & \tau_{k-1}^{(\alpha)}/z \\ \tau_{k+1}^{(\alpha)}/z & \tau_k^{(\alpha)} \end{bmatrix},$$

where $S^\pm(z) = (1 - S/z)^{\pm 1}$ are the shift fields, constructed from the elementary shift $S : \mathbb{C}[c_k] \rightarrow \mathbb{C}[c_k]$, defined as the multiplicative map such that $S(1) = 0, S(c_k) = c_{k+1}$. The shift fields $S^\pm(z)$ play a similar role here as the vertex operator $\Gamma(z)$ does in the theory of the KdV hierarchy.

Next we introduce linear equations for the 2×2 Baker functions:

$$\Psi_k^{(\alpha+1)} = \Psi_k^{(\alpha)} V_k^{(\alpha)}, \quad \Psi_{k-1}^{(\alpha+1)} = \Psi_k^{(\alpha)} W_k^{(\alpha)}.$$

Then we show that compatibility of these equations leads to the discrete zero curvature equations

$$V_k^{(\alpha)} (W_{k+1}^{(\alpha)})^{-1} = (W_{k+1}^{(\alpha-1)})^{-1} V_{k+1}^{(\alpha-1)}.$$

Since we can give explicit expressions for the connection matrices $V_k^{(\alpha)}, W_k^{(\alpha)}$ in terms of the τ -functions we obtain the following basic equation:

$$(\tau_k^{(\alpha)})^2 = \tau_k^{(\alpha-1)} \tau_k^{(\alpha+1)} - \tau_{k+1}^{(\alpha-1)} \tau_{k-1}^{(\alpha+1)}, \quad k = 0, 1, \dots$$

This is upon a change of variables the Q -system, see e.g., [5].

In the second part of the paper we generalize our derivation of the Q -system by using the loop group of \mathbf{GL}_3 , obtaining τ -functions $\tau_{k,l}^{(\alpha,\beta)}(c_k, d_k, e_k)$, where $k, l, \alpha, \beta \in \mathbb{Z}$ and the c_k, d_k, e_k are coordinates on the lower triangular subgroup of the loop group of \mathbf{GL}_3 . We can explicitly calculate these τ -functions, see Theorem 3.1, but the answer is much more complicated than the simple Hankel determinants in the 2×2 case. Next we introduce Baker functions, now 3×3 matrices depending on a spectral parameter, and the linear equations for the Baker functions. Again we can explicitly calculate the connection matrices in terms of τ -functions, see Lemma 3.6 and Lemma 3.7. Compatibility now gives us a system of *four* equations (Theorem 3.8)

$$(1.3) \quad \begin{cases} \tau_{k,l-1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha+1,\beta)} + \tau_{k+1,l}^{(\alpha,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} - \tau_{k,l-1}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)} = 0, \\ \tau_{k+1,l+1}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)} - \tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha,\beta+1)} + \tau_{k+1,l}^{(\alpha,\beta+1)} \tau_{k,l+1}^{(\alpha,\beta)} = 0, \\ (\tau_{k,l}^{(\alpha,\beta)})^2 = \tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha-1,\beta)} + \tau_{k+1,l+1}^{(\alpha-1,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} - \tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k-1,l}^{(\alpha+1,\beta)}, \\ (\tau_{k,l}^{(\alpha,\beta)})^2 = \tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta-1)} - \tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l-1}^{(\alpha,\beta+1)} - \tau_{k-1,l}^{(\alpha,\beta-1)} \tau_{k+1,l}^{(\alpha,\beta+1)}. \end{cases}$$

The first two of these new equations are generalizations of T -system equations, more precisely the first equation is for fixed β a T -system, after a change of variables, and for the second equation one fixes α . See e.g., [10] for information on T -systems.

The last two equations seem to be related to the fusion rules of quantum transfer matrices for Young diagrams consisting of two rectangular blocks, see [12], [11].

2. THE 2×2 CASE

2.1. 2×2 τ -functions and Q -system. We have an action of the central extension $\widehat{\mathrm{GL}}_2$ of the loop group $\widetilde{\mathrm{GL}}_2 = \mathrm{GL}_2(\mathbb{C}((z^{-1})))$ on two component semi-infinite wedge space $F^{(2)}$, see Appendix A.4. Let $\pi: \widehat{\mathrm{GL}}_2 \rightarrow \widetilde{\mathrm{GL}}_2$ be the projection on the loop group and consider the action on the vacuum vector of $F^{(2)}$, see (A.4), of $g_C \in \widehat{\mathrm{GL}}_2$, where

$$(2.1) \quad \pi(g_C) = \begin{bmatrix} 1 & 0 \\ C(z) & 1 \end{bmatrix}.$$

Here

$$C(z) = \sum_{i \in \mathbb{Z}} c_i z^{-i-1},$$

where $c_i \in \mathbb{C}$; sometimes it is also useful to think of the c_i as formal variables.

Recall (see (A.12)) the projection of the fermionic translation operators to the loop group $\widetilde{\mathrm{GL}}_2$: in particular we get the matrices

$$\pi(Q_0) = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi(Q_1) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

We also have the translation element

$$T = Q_1 Q_0^{-1} \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

We define shifts on the series $C(z)$ by

$$(2.2) \quad C^{(\alpha)}(z) = z^\alpha C(z) = \sum_{i \in \mathbb{Z}} c_{i+\alpha} z^{-i-1}.$$

We define similarly the shifted group element $g^{(\alpha)} = Q_0^\alpha g_C Q_0^{-\alpha}$ so that

$$(2.3) \quad \pi(g^{(\alpha)}) = \begin{bmatrix} 1 & 0 \\ C^{(\alpha)}(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z^\alpha C(z) & 1 \end{bmatrix}.$$

We then have

$$(2.4) \quad Q_0^{-1} g_C^{(\alpha+1)} = g_C^{(\alpha)} Q_0^{-1},$$

and the same relation with π applied,

$$(2.5) \quad \pi(Q_0)^{-1} \pi(g_C^{(\alpha+1)}) = \pi(g_C^{(\alpha)}) \pi(Q_0)^{-1}.$$

The fundamental objects in the theory of the Toda lattice and Q -systems are the τ -functions defined by

$$(2.6) \quad \boxed{\tau_k^{(\alpha)} = \langle T^k v_0, g_C^{(\alpha)} v_0 \rangle.}$$

The τ -functions in the 2×2 case are determinants of Hankel matrices.

Theorem 2.1.

$$\begin{aligned} \tau_k^{(\alpha)} &= \frac{1}{k!} \text{Res}_{\mathbf{w}} \left(\prod_{i=1}^k C^{(\alpha)}(w_i) \prod_{k \geq j > i \geq 0} (w_i - w_j)^2 \right) = \\ &= \det \begin{bmatrix} c_\alpha & c_{\alpha+1} & \cdots & c_{\alpha+k-1} \\ c_{\alpha+1} & c_{\alpha+2} & \cdots & c_{\alpha+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\alpha+k-1} & c_{\alpha+k} & \cdots & c_{\alpha+2k-2} \end{bmatrix}. \end{aligned}$$

For the proof see Appendix B.1.

The simple form of the τ -functions allows us to apply the Desnanot-Jacobi identity (cf., [3]) to obtain ([5]) the following difference equations, satisfied by our tau functions:

$$(2.7) \quad \tau_k^{(\alpha)} \tau_{k-2}^{(\alpha+2)} = \tau_{k-1}^{(\alpha+2)} \tau_{k-1}^{(\alpha)} - (\tau_{k-1}^{(\alpha+1)})^2.$$

The disadvantage of obtaining the difference equations in this way is that it is not at all apparent how to generalize to the 3×3 situation, in which the formulas for the tau functions are much more complicated. We thus present another way of obtaining our 2×2 difference equations.

2.2. Birkhoff Factorization. Define an element of the central extension of the loop group of \mathbf{GL}_2 :

$$(2.8) \quad g^{[k](\alpha)} = T^{-k} g_C^{(\alpha)},$$

and assume that it has a Birkhoff Factorization ([14], see also Appendix C.1):

$$g^{[k](\alpha)} = g_-^{[k](\alpha)} g_{0,+}^{[k](\alpha)},$$

where $\pi(g_-^{[k](\alpha)}) = 1 + \mathcal{O}(z^{-1})$ and $\pi(g_{0,+}^{[k](\alpha)}) = A_k^{(\alpha)} + \mathcal{O}(z)$, for $A_k^{(\alpha)}$ an invertible z independent matrix. This assumption is justified precisely when $\tau_k^{(\alpha)}(g) = \langle v_0, g^{[k](\alpha)} v_0 \rangle$ is not zero.

Now we want display the negative component of $\pi(g^{[k](\alpha)})$. To calculate this we make some extra structure explicit.

Let \mathcal{N} be the subgroup of elements of $\widetilde{\text{SL}}_2$ of the form (2.1). We can think of the coefficients c_k as coordinates on \mathcal{N} , so

$$B = \mathbb{C}[c_k]_{k \in \mathbb{Z}}$$

is the coordinate ring of \mathcal{N} .

Define first shifts acting on B : these are multiplicative maps given on generators by

$$(2.9) \quad S^\alpha: B \rightarrow B, \quad S^\alpha(1) = 0, \quad S^\alpha(c_k) = c_{k+\alpha}, \quad \alpha \in \mathbb{Z}.$$

We will often write S^\pm for $S^{\pm 1}$.

Define also *shift fields*. These are multiplicative maps

$$(2.10) \quad S^\pm(z): B \rightarrow B[[z^{-1}]],$$

given by

$$S^\pm(z) = \left(1 - \frac{S^+}{z}\right)^{\pm 1},$$

Theorem 2.2. For $k \geq 0$ and all $\alpha \in \mathbb{Z}$

$$\pi(g_-^{[k](\alpha)}) = \begin{bmatrix} S^+(z)\tau_k^{(\alpha)} & S^+(z)\tau_{k-1}^{(\alpha)}/z \\ S^-(z)\tau_{k+1}^{(\alpha)}/z & S^-(z)\tau_k^{(\alpha)} \end{bmatrix} / \tau_k^{(\alpha)}.$$

We sketch the proof in Appendix C.2.

Now we have expressed the negative component of the Birkhoff factorization in terms of matrix elements of the centrally extended loop group we have no longer need for the central extension and we will simplify notation: we will write $g_-^{[k](\alpha)}$ for $\pi(g_-^{[k](\alpha)})$, T for $\pi(T)$ and similarly Q_a for $\pi(Q_a)$, $a = 0, 1$. In particular in the rest of this section we write

$$T = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} = Q_1 Q_0^{-1}, \quad Q_0 = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

2.3. Matrix Baker Functions and Connection Matrices. Next we define the Baker functions. These are elements of the loop group defined by

$$(2.11) \quad \Psi^{[k](\alpha)} = T^k Q_0^{-\alpha} g_-^{[k](\alpha)}.$$

Since the Baker functions are all invertible, they are related by *connection matrices* belonging to the loop group: define $\Gamma_{[k],(\alpha)}^{[l],(\beta)} \in \widetilde{\text{GL}}_2$ by

$$\Psi^{[l](\beta)} = \Psi^{[k](\alpha)} \Gamma_{[k],(\alpha)}^{[l],(\beta)}.$$

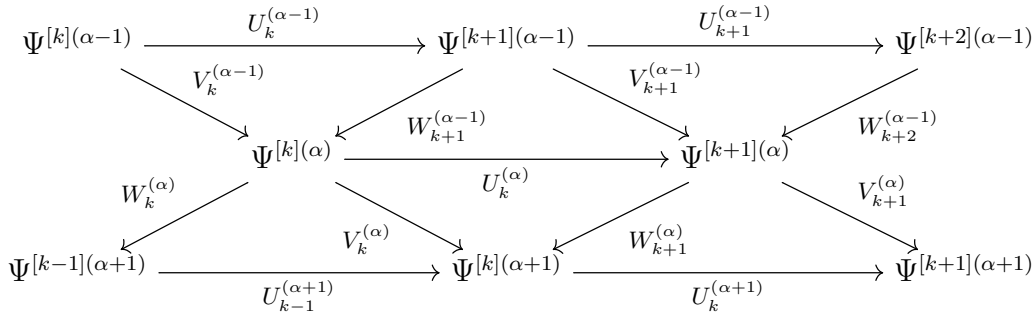
We are in particular interested in connection matrices that implement nearest neighbor steps on the lattice of Baker functions. So define

$$U_k^{(\alpha)} = \Gamma_{[k](\alpha)}^{[k+1](\alpha)}, \quad V_k^{(\alpha)} = \Gamma_{[k](\alpha)}^{[k](\alpha+1)}, \quad W_k^{(\alpha)} = \Gamma_{[k](\alpha)}^{[k-1](\alpha+1)},$$

so that we have

$$(2.12) \quad \begin{aligned} \Psi^{[k+1](\alpha)} &= \Psi^{[k](\alpha)} U_k^{(\alpha)}, \\ \Psi^{[k](\alpha+1)} &= \Psi^{[k](\alpha)} V_k^{(\alpha)}, \\ \Psi^{[k-1](\alpha+1)} &= \Psi^{[k](\alpha)} W_k^{(\alpha)}. \end{aligned}$$

Pictorially:



Walking around the triangles in this diagram we see that we get factorizations of all basic connection matrices. In particular

$$(2.13) \quad U_k^{(\alpha)} = V_k^{(\alpha)} (W_{k+1}^{(\alpha)})^{-1} = (W_{k+1}^{(\alpha-1)})^{-1} V_{k+1}^{(\alpha-1)}.$$

Such factorizations are well known in the theory of integrable systems, see for instance Adler[1], Sklyanin[15].

We study the connection matrices more explicitly.

Lemma 2.3.

$$(2.14) \quad U_k^{(\alpha)} = (g_-^{[k](\alpha)})^{-1} T g_-^{[k+1](\alpha)} = g_{0+}^{[k](\alpha)} (g_{0+}^{[k+1](\alpha)})^{-1},$$

$$(2.15) \quad V_k^{(\alpha)} = (g_-^{[k](\alpha)})^{-1} Q_0^{-1} g_-^{[k](\alpha+1)} = g_{0+}^{[k](\alpha)} Q_0^{-1} (g_{0+}^{[k](\alpha+1)})^{-1},$$

$$(2.16) \quad W_k^{(\alpha)} = (g_-^{[k](\alpha)})^{-1} Q_1^{-1} g_-^{[k-1](\alpha+1)} = g_{0+}^{[k](\alpha)} Q_0^{-1} (g_{0+}^{[k-1](\alpha+1)})^{-1}.$$

Proof. The first expression for the connection matrices (in terms of negative components $g_-^{[k](\alpha)}$) follows from the definition (2.12) of the connection matrices and the definition (2.11) of the Baker functions. It also uses $Q_0 T = Q_1$.

For the second expression for the connection matrices in term of positive components $g_{0+}^{[k](\alpha)}$ we use (see also (2.4), or rather (2.5))

$$\begin{aligned} T g_-^{[k+1](\alpha)} g_{0+}^{[k+1](\alpha)} &= g_-^{[k](\alpha)} g_{0+}^{[k](\alpha)}, \\ Q_0^{-1} g_-^{[k](\alpha+1)} g_{0+}^{[k](\alpha+1)} &= g_-^{[k](\alpha)} g_{0+}^{[k](\alpha)} Q_0^{-1}, \\ Q_1^{-1} g_-^{[k-1](\alpha+1)} g_{0+}^{[k-1](\alpha+1)} &= g_-^{[k](\alpha)} g_{0+}^{[k](\alpha)} Q_0^{-1}. \end{aligned}$$

Rearranging factors proves then the second form for the connection matrices. \square

Note that the second equality in the above Lemma tells us that the elementary connection matrices are positive in z , i.e., contain only z^k for $k \geq 0$. This allows us to easily calculate the connection matrices in terms of τ -functions.

First note that Theorem 2.2 allows us to expand $g^{[k](\alpha)}$ and its inverse up to order z^{-1} as

$$(2.17) \quad \begin{aligned} g_-^{[k](\alpha)} &= 1_{2 \times 2} + \frac{1}{z} \begin{bmatrix} S^+[1] \tau_k^{(\alpha)} / \tau_k^{(\alpha)} & \frac{1}{h_{k-1}^{(\alpha)}} \\ h_k^{(\alpha)} & S^-[1] \tau_k^{(\alpha)} / \tau_k^{(\alpha)} \end{bmatrix} + \mathcal{O}(z^{-2}), \\ (g_-^{[k](\alpha)})^{-1} &= 1_{2 \times 2} + \frac{1}{z} \begin{bmatrix} -S^+[1] \tau_k^{(\alpha)} / \tau_k^{(\alpha)} & \frac{-1}{h_{k-1}^{(\alpha)}} \\ -h_k^{(\alpha)} & -S^-[1] \tau_k^{(\alpha)} / \tau_k^{(\alpha)} \end{bmatrix} + \mathcal{O}(z^{-2}). \end{aligned}$$

Here

$$h_k^{(\alpha)} = \frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}},$$

and we expand the shift fields in partial shifts:

$$S^\pm(z) f = \sum_{n=0}^{\infty} S^\pm[n] f z^{-n},$$

Lemma 2.4. *We have*

$$U_k^{(\alpha)} = \begin{bmatrix} z - b_k^{(\alpha)} & \frac{1}{h_k^{(\alpha)}} \\ -h_k^{(\alpha)} & 0 \end{bmatrix}.$$

Similarly,

$$V_k^{(\alpha)} = \begin{bmatrix} z - \alpha_{k-1}^{(\alpha)} & \frac{1}{h_{k-1}^{(\alpha+1)}} \\ -h_k^{(\alpha)} & 1 \end{bmatrix}, \quad W_k^{(\alpha)} = \begin{bmatrix} 1 & -\frac{1}{h_{k-1}^{(\alpha)}} \\ h_{k-1}^{(\alpha+1)} & z - \beta_{k-1}^{(\alpha)} \end{bmatrix}, \quad k = 0, 1, \dots$$

Here

$$\alpha_k^{(\alpha)} = \frac{h_{k+1}^{(\alpha)}}{h_k^{(\alpha+1)}}, \quad \beta_k^{(\alpha)} = \frac{h_k^{(\alpha+1)}}{h_k^{(\alpha)}}, \quad b_k^{(\alpha)} = \alpha_{k-1}^{(\alpha)} + \beta_{k-1}^{(\alpha)},$$

and we put $\alpha_{-1}^{(\alpha)} = \frac{1}{h_{-1}^{(\alpha)}} = 0$.

Note that each $U_k^{(\alpha)}$ has determinant 1, and that $\det(V_k^{(\alpha)}) = z = \det(W_k^{(\alpha)})$.

Proof. As an example we calculate $V_k^{(\alpha)} = \left(g_-^{[k](\alpha)}\right)^{-1} Q_0^{-1} g^{[k](\alpha+1)}$, using (2.17):

$$\begin{aligned} V_k^{(\alpha)} &= \begin{bmatrix} 1 + x/z + \mathcal{O}(z^{-2}) & \mathcal{O}(z^{-1}) \\ -\frac{h_k^{(\alpha)}}{z} + \mathcal{O}(z^{-2}) & 1 + \mathcal{O}(z^{-1}) \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 + y/z + \mathcal{O}(z^{-2}) & \frac{1}{zh_{k-1}^{(\alpha+1)}} \\ \mathcal{O}(z^{-1}) & 1 + \mathcal{O}(z^{-1}) \end{bmatrix} = \\ &= \begin{bmatrix} z + x + y & \frac{1}{h_{k-1}^{(\alpha+1)}} \\ -h_k^{(\alpha)} & 1 \end{bmatrix}, \end{aligned}$$

dropping all terms containing z^{-1} or lower. Here x, y are some expressions in the τ -functions we will determine by noting that $\det(V_k^{(\alpha)}) = z$. We see that $x + y = \frac{-h_k^{(\alpha)}}{h_{k-1}^{(\alpha+1)}} = -\alpha_{k-1}^{(\alpha)}$. This proves the Lemma for $V_k^{(\alpha)}$, $k = 1, 2, \dots$. The proof for $W_k^{(\alpha)}$ and $V_0^{(\alpha)}$ is similar. To get the formula for $U_k^{(\alpha)}$ use $U_k^{(\alpha)} = V_k^{(\alpha)} \left(W_{k+1}^{(\alpha)}\right)^{-1}$, again discarding terms involving z^{-1} and lower. \square

Now we return to the two factorization (2.13) of the connection matrix. Using the expressions for $V_k^{(\alpha)}, W_k^{(\alpha)}$ from Lemma 2.4 we find two expressions for the coefficient $b_k^{(\alpha)}$ in $U_k^{(\alpha)}$:

$$b_k^{(\alpha)} = \alpha_{k-1}^{(\alpha)} + \beta_{k-1}^{(\alpha)} = \alpha_k^{(\alpha-1)} + \beta_k^{(\alpha-1)},$$

giving equations for the $h_k^{(\alpha)}$ variables:

$$(2.18) \quad \frac{h_k^{(\alpha)}}{h_{k-1}^{(\alpha+1)}} + \frac{h_{k-1}^{(\alpha+1)}}{h_{k-1}^{(\alpha)}} = \frac{h_{k+1}^{(\alpha-1)}}{h_k^{(\alpha)}} + \frac{h_k^{(\alpha)}}{h_k^{(\alpha-1)}}.$$

Theorem 2.5. *The equations (2.18) are equivalent to the Q-system*

$$(2.19) \quad (\tau_k^{(\alpha)})^2 = \tau_k^{(\alpha-1)} \tau_k^{(\alpha+1)} - \tau_{k+1}^{(\alpha-1)} \tau_{k-1}^{(\alpha+1)}, \quad k = 0, 1, \dots$$

Proof. Write (2.18) out in terms of τ -functions:

$$\frac{\tau_{k+1}^{(\alpha)} \tau_{k-1}^{(\alpha+1)}}{\tau_k^{(\alpha)} \tau_k^{(\alpha+1)}} + \frac{\tau_k^{(\alpha+1)} \tau_{k-1}^{(\alpha)}}{\tau_{k-1}^{(\alpha+1)} \tau_k^{(\alpha)}} = \frac{\tau_{k+2}^{(\alpha-1)} \tau_k^{(\alpha)}}{\tau_{k+1}^{(\alpha-1)} \tau_{k+1}^{(\alpha)}} + \frac{\tau_{k+1}^{(\alpha)} \tau_k^{(\alpha-1)}}{\tau_k^{(\alpha)} \tau_{k+1}^{(\alpha-1)}}.$$

Writing all terms under the same denominator and rearranging terms, we see that this is equivalent to:

$$(2.20) \quad (\tau_k^{(\alpha)})^2 (\tau_{k+2}^{(\alpha-1)} \tau_k^{(\alpha+1)} - \tau_{k+1}^{(\alpha-1)} \tau_{k+1}^{(\alpha+1)}) = (\tau_{k+1}^{(\alpha)})^2 (\tau_{k+1}^{(\alpha-1)} \tau_{k-1}^{(\alpha+1)} - \tau_k^{(\alpha-1)} \tau_k^{(\alpha+1)}).$$

Notice that if

$$(\tau_k^{(\alpha)})^2 = \tau_k^{(\alpha-1)} \tau_k^{(\alpha+1)} - \tau_{k+1}^{(\alpha-1)} \tau_{k-1}^{(\alpha+1)},$$

then (2.20) implies

$$(\tau_{k+1}^{(\alpha)})^2 = \tau_{k+1}^{(\alpha-1)} \tau_{k+1}^{(\alpha+1)} - \tau_{k+2}^{(\alpha-1)} \tau_k^{(\alpha+1)}.$$

We thus need only prove that the equality holds for $k = 0$. But this is just

$$(\tau_0^\alpha)^2 = \tau_0^{(\alpha-1)} \tau_0^{(\alpha+1)} - \tau_1^{(\alpha-1)} \tau_{-1}^{(\alpha+1)},$$

which is true since $\tau_{-1}^{(\alpha)} = 0$ and $\tau_0^{(\alpha)} = 1$ for all α . So the theorem follows. \square

So we have rederived the equations (2.7) using the Birkhoff factorization.

3. 3×3 CASE

3.1. τ -functions. We proceed very much as in the 2×2 case.

We have an action of the central extension $\widetilde{\text{GL}}_3$ of the loop group $\widetilde{\text{GL}}_3 = \text{GL}_3(\mathbb{C}((z^{-1})))$ on three component semi-infinite wedge space $F^{(3)}$, see Appendix A.4. Let $\pi: \widetilde{\text{GL}}_3 \rightarrow \widetilde{\text{GL}}_3$ be the projection on the loop group and consider the action on the vacuum vector of $F^{(3)}$, see (A.4), of $g_C \in \widetilde{\text{GL}}_2$, where

$$(3.1) \quad \pi(g_C) = \begin{bmatrix} 1 & 0 & 0 \\ C(z) & 1 & 0 \\ D(z) & E(z) & 1 \end{bmatrix}.$$

Here

$$X(z) = \sum_{i \in \mathbb{Z}} x_i z^{-i-1}, \quad X = C, D, E, \quad x = c, d, e.$$

where $x_i \in \mathbb{C}$; sometimes it is also useful to think of the x_i as formal variables.

Recall (see (A.12)) the projection of the fermionic translation operators to the loop group $\widetilde{\text{GL}}_3$: in particular we get the (commuting) matrices

$$\pi(Q_0) = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi(Q_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi(Q_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}.$$

We also have the translation elements

$$T_1 = Q_1 Q_0^{-1} \xrightarrow{\pi} \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = Q_2 Q_1^{-1} \xrightarrow{\pi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}$$

We define shifts on the series $X(z)$ by

$$(3.2) \quad X^{(\alpha)}(z) = z^\alpha X(z) = \sum_{i \in \mathbb{Z}} x_{i+\alpha} z^{-i-1}, \quad X = C, D, E, \quad x = c, d, e.$$

We define similarly the shifted group element $g^{(\alpha, \beta)} = Q_0^\alpha Q_1^\beta g_C Q_1^{-\beta} Q_0^{-\alpha}$ so that

$$(3.3) \quad \pi(g_C^{(\alpha, \beta)}) = \begin{bmatrix} 1 & 0 & 0 \\ C^{(\alpha-\beta)}(z) & 1 & 0 \\ D^{(\alpha)}(z) & E^{(\beta)}(z) & 1 \end{bmatrix}.$$

We then have (using $Q_0 Q_1 = -Q_1 Q_0$ twice)

$$(3.4) \quad Q_0^{-1} g^{(\alpha+1, \beta)} = g^{(\alpha, \beta)} Q_0^{-1}, \quad Q_1^{-1} g^{(\alpha, \beta+1)} = g^{(\alpha, \beta)} Q_1^{-1},$$

and the same relations with π applied,

$$(3.5) \quad \pi(Q_0)^{-1} \pi(g^{(\alpha+1, \beta)}) = \pi(g^{(\alpha, \beta)}) \pi(Q_0)^{-1}, \quad \pi(Q_1)^{-1} \pi(g^{(\alpha, \beta+1)}) = \pi(g^{(\alpha, \beta)}) \pi(Q_1)^{-1},$$

The fundamental objects in the 3×3 theory are the τ -functions defined by

$$(3.6) \quad \boxed{\tau_{k,l}^{(\alpha,\beta)} = \langle T_1^k T_2^l v_0, g^{(\alpha,\beta)} v_0 \rangle .}$$

Note that if we introduce another translation group element $T_3 = Q_2 Q_0^{-1}$ then we can write non uniquely

$$T_1^k T_2^l = (\pm 1) T_1^{n_c} T_2^{n_e} T_3^{n_d} ,$$

where $k = n_c + n_d$, $l = n_d + n_e$, and we take $n_c, n_d, n_e \geq 0$.

Theorem 3.1.

$$(3.7) \quad \tau_{k,l}^{\alpha,\beta} = \sum_{\substack{n_c+n_d=k, n_d+n_e=l \\ n_c, n_d, n_e \geq 0}} c_{n_c, n_d, n_e}^{(\alpha,\beta)}$$

where

$$c_{n_c, n_d, n_e}^{(\alpha,\beta)} = \frac{(-1)^{\frac{n_d(n_d+1)}{2}}}{n_c! n_d! n_e!} \text{Res}_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \left(\prod_{i=1}^{n_c} C^{(\alpha-\beta)}(x_i) \prod_{i=1}^{n_d} D^{(\alpha)}(y_i) \prod_{i=1}^{n_e} E^{(\beta)}(z_i) p_{n_c, n_d, n_e} \right) ,$$

and

$$\begin{aligned} p_{n_c, n_c, n_e} &= \\ &= \frac{\prod_{1 \leq i < j \leq n_c} (x_i - x_j)^2 \prod_{1 \leq i < j \leq n_d} (y_i - y_j)^2 \prod_{1 \leq i < j \leq n_e} (z_i - z_j)^2 \prod_{i=1}^{n_c} \prod_{j=1}^{n_d} (x_i - y_j) \prod_{i=1}^{n_d} \prod_{j=1}^{n_e} (y_i - z_j)}{\prod_{i=1}^{n_c} \prod_{j=1}^{n_e} (x_i - z_j)} \end{aligned}$$

We discuss the proof in Appendix B.2.

3.2. Examples of 3×3 tau functions.

$$(1) \quad \tau_{k,\ell}^{(\alpha,\beta)} = 0 \text{ if } k < 0 \text{ or } \ell < 0$$

$$(2) \quad \tau_{0,0}^{(\alpha,\beta)} = 1$$

$$(3) \quad \tau_{k,0}^{(\alpha,\beta)} = \det \begin{bmatrix} c_{\alpha-\beta} & c_{\alpha-\beta+1} & \cdots & c_{\alpha-\beta+k-1} \\ c_{\alpha-\beta+1} & c_{\alpha-\beta+2} & \cdots & c_{\alpha-\beta+k} \\ \vdots & \vdots & \cdots & \vdots \\ c_{\alpha-\beta+k-1} & c_{\alpha-\beta+k} & \cdots & c_{\alpha-\beta+2k-2} \end{bmatrix}$$

$$(4) \quad \tau_{0,k}^{(\alpha,\beta)} = \det \begin{bmatrix} e_{\beta} & e_{\beta+1} & \cdots & e_{\beta+k-1} \\ e_{\beta+1} & e_{\beta+2} & \cdots & e_{\beta+k} \\ \vdots & \vdots & \cdots & \vdots \\ e_{\beta+k-1} & e_{\beta+k} & \cdots & e_{\beta+2k-2} \end{bmatrix}$$

$$(5) \quad \tau_{1,1}^{(\alpha,\beta)} = -d_{\alpha} + \sum_{i=0}^{\infty} e_{\beta+i} c_{\alpha-\beta-i-1}$$

$$(6) \quad \tau_{1,2}^{(\alpha,\beta)} = e_{\beta} \sum_{i=1}^{\infty} e_{\beta+i+1} c_{\alpha-\beta-i-1} - e_{\beta+1} \sum_{i=0}^{\infty} e_{\beta+i+1} c_{\alpha-\beta-i-2} + e_{\beta+1} d_{\alpha} - e_{\beta} d_{\alpha+1}$$

$$(7) \quad \tau_{2,1}^{(\alpha,\beta)} = c_{\alpha-\beta+1} \sum_{i=0}^{\infty} e_{\beta+i} c_{\alpha-\beta-i-1} - c_{\alpha-\beta} \sum_{i=0}^{\infty} e_{\beta+i} c_{\alpha-\beta-i} + c_{\alpha-\beta} d_{\alpha+1} - c_{\alpha-\beta+1} d_{\alpha}$$

Remark 3.2. Note that the summands $c_{n_c, n_d, n_e}^{(\alpha, \beta)}$ of the τ -functions are of degree n_x in the coefficients x_k of the series $X(z) = \sum x_k z^{-k-1}$, for $x = c, d, e$, $X = C, D, E$.

3.3. Birkhoff Factorization for 3×3 . Define centrally extended loop group elements

$$(3.8) \quad g^{[k, l](\alpha, \beta)} = T_2^{-l} T_1^{-k} g^{(\alpha, \beta)},$$

and assume that they have a Birkhoff Factorization([14], see also Appendix C.1):

$$g^{[k, l](\alpha, \beta)} = g_-^{[k, l](\alpha, \beta)} g_{0,+}^{[k, l](\alpha, \beta)},$$

where $\pi(g_-^{[k, l](\alpha, \beta)}) = 1 + \mathcal{O}(z^{-1})$ and $\pi(g_{0,+}^{[k, l](\alpha, \beta)}) = A_{k,l}^{(\alpha, \beta)} + \mathcal{O}(z)$, for $A_{k,l}^{(\alpha, \beta)}$ an invertible z independent matrix. This assumption is justified precisely when $\tau_{k,l}^{(\alpha, \beta)}(g) = \langle v_0, g^{[k, l](\alpha, \beta)} v_0 \rangle$ is not zero.

Now we want display the negative component of $\pi(g^{[k, l](\alpha, \beta)})$. To calculate this we make some extra structure explicit.

Let \mathcal{N} be the subgroup of elements of $\widetilde{\text{SL}}_3$ of the form (3.3). We can think of the coefficients x_k , $x = c, d, e$ as coordinates on \mathcal{N} , so

$$B = \mathbb{C}[[c_k, d_k, e_k]]_{k \in \mathbb{Z}}$$

is the coordinate ring of \mathcal{N} .

Define first shifts acting on B : these are multiplicative maps given on generators by $(x, y = c, d, e)$

$$(3.9) \quad S_x^\alpha: B \rightarrow B, \quad S_x^\alpha(1) = 0, \quad S_x^\alpha(x_k) = x_{k+\alpha}, \quad S_x^\alpha(y_k) = y_k, y \neq x, \quad \alpha \in \mathbb{Z}.$$

We will often write S_x^\pm for $S_x^{\pm 1}$.

Define also *shift fields*. These are multiplicative maps

$$(3.10) \quad S_x^\pm(z): B \rightarrow B[[z^{-1}]],$$

given by

$$S_x^\pm(z) = \left(1 - \frac{S_x^+}{z}\right)^{\pm 1},$$

Theorem 3.3. For $k \geq 0$ and all $\alpha \in \mathbb{Z}$

$$\pi(g_-^{[k, l](\alpha, \beta)}) = \left(\Sigma \mathcal{T}_{k,l}^{(\alpha, \beta)}\right) / \tau_{kl}^{(\alpha, \beta)},$$

where

$$\Sigma = \begin{bmatrix} S_c^+(z) S_d^+(z) & 0 & 0 \\ 0 & S_c^-(z) S_e^+(z) & 0 \\ 0 & 0 & S_d^-(z) S_e^-(z) \end{bmatrix},$$

$$\mathcal{T}_{k,l}^{(\alpha, \beta)} = \begin{bmatrix} \tau_{k,l}^{(\alpha, \beta)} & \frac{\tau_{k-1,l}^{(\alpha, \beta)}}{z} & (-1)^k \frac{\tau_{k-1,l-1}^{(\alpha, \beta)}}{z} \\ \frac{\tau_{k+1,l}^{(\alpha, \beta)}}{z} & \tau_{k,l}^{(\alpha, \beta)} & (-1)^k \frac{\tau_{k,l-1}^{(\alpha, \beta)}}{z} \\ (-1)^{k+1} \frac{\tau_{k+1,l+1}^{(\alpha, \beta)}}{z} & (-1)^k \frac{\tau_{k,l+1}^{(\alpha, \beta)}}{z} & \tau_{k,l}^{(\alpha, \beta)} \end{bmatrix}$$

We sketch the proof in Appendix C.3

Now we have expressed the negative component of the Birkhoff factorization in terms of matrix elements of the centrally extended loop group we have no longer need for the central extension and we will simplify notation: we will write $g_-^{[k,l](\alpha,\beta)}$ for $\pi(g_-^{[k,l](\alpha,\beta)})$, T_i for $\pi(T_i)$, $i = 1, 2$ and similarly Q_a for $\pi(Q_a)$, $a = 0, 1, 3$. In particular in the rest of this section we write

$$T_1 = \begin{bmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = Q_1 Q_0^{-1}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^{-1} \end{bmatrix} = Q_2 Q_1^{-1},$$

and

$$Q_0 = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}.$$

3.4. Matrix Baker Functions and Connection Matrices, 3×3 case. Next we define the Baker functions: these are now elements of the loop group of \mathbf{GL}_3 defined by

$$\Psi^{[k,l](\alpha,\beta)} = T_1^k T_2^l Q_0^{-\alpha} Q_1^{-\beta} g_-^{[k,l](\alpha,\beta)}.$$

Since the Baker functions are all invertible, they are related by (right) multiplication by connection matrices belonging to $\widetilde{\mathbf{GL}}_3$. In particular, define

$$(3.11) \quad \Psi^{[k',l'](\alpha',\beta')} = \Psi^{[k,l](\alpha,\beta)} \Gamma_{[k,l](\alpha,\beta)}^{[k',l'](\alpha',\beta')},$$

so that

$$\Gamma_{[k,l](\alpha,\beta)}^{[k',l'](\alpha',\beta')} = (g_-^{[k,l](\alpha,\beta)})^{-1} Q_0^{x_0} Q_1^{x_1} Q_2^{x_2} g_-^{[k',l'](\alpha',\beta')},$$

where

$$\begin{aligned} x_0 &= k - k' + \alpha - \alpha', \\ x_1 &= k' - k + l - l' + \beta - \beta', \\ x_2 &= l' - l. \end{aligned}$$

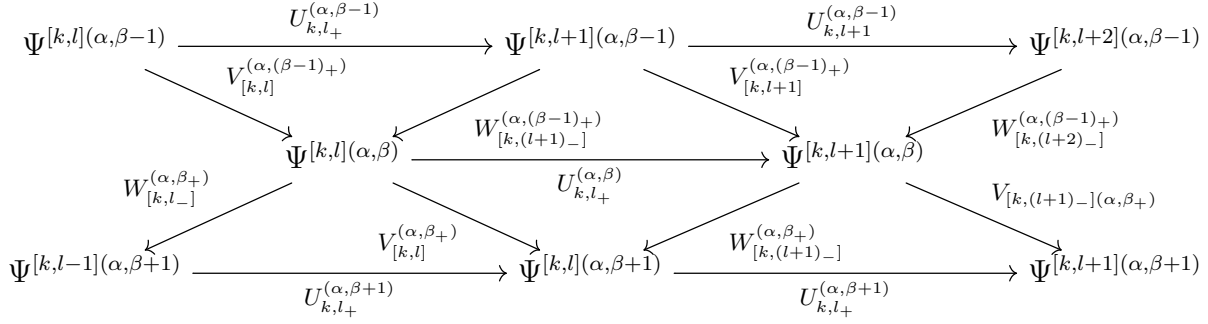
The simplest connection matrices are those where (x_0, x_1, x_2) has two zero components and the other absolute value 1. Define therefore the elementary connection matrices

$$(3.12) \quad \begin{aligned} V_{[k,l]}^{(\alpha, \beta)} &= \Gamma_{[k,l](\alpha,\beta)}^{[k,l](\alpha+1,\beta)} = (g_-^{[k,l](\alpha,\beta)})^{-1} Q_0^{-1} g_-^{[k,l](\alpha+1,\beta)}, \\ V_{[k,l]}^{(\alpha, \beta+)} &= \Gamma_{[k,l](\alpha,\beta)}^{[k,l](\alpha,\beta+1)} = (g_-^{[k,l](\alpha,\beta)})^{-1} Q_1^{-1} g_-^{[k,l](\alpha,\beta+1)}, \\ W_{[k-,l]}^{(\alpha, \beta)} &= \Gamma_{[k,l](\alpha,\beta)}^{[k-1,l](\alpha+1,\beta)} = (g_-^{[k,l](\alpha,\beta)})^{-1} Q_1^{-1} g_-^{[k-1,l](\alpha+1,\beta)}, \\ W_{[k,l-]}^{(\alpha, \beta+)} &= \Gamma_{[k,l](\alpha,\beta)}^{[k,l-1](\alpha,\beta+1)} = (g_-^{[k,l](\alpha,\beta)})^{-1} Q_2^{-1} g_-^{[k,l-1](\alpha,\beta+1)}. \end{aligned}$$

Also define translation matrices

$$(3.13) \quad \begin{aligned} U_{[k+,l]}^{(\alpha,\beta)} &= \Gamma_{[k,l](\alpha,\beta)}^{[k+1,l](\alpha,\beta)} = (g_-^{[k,l](\alpha,\beta)})^{-1} T_1 (g_-^{[k+1,l](\alpha,\beta)}), \\ U_{[k,l+]}^{(\alpha,\beta)} &= \Gamma_{[k,l](\alpha,\beta)}^{[k,l+1](\alpha,\beta)} = (g_-^{[k,l](\alpha,\beta)})^{-1} T_2 (g_-^{[k,l+1](\alpha,\beta)}). \end{aligned}$$

Pictorially, fixing k, α



Walking around the triangles in this and the similar diagram where l, β are fixed we see that we get factorizations of the basic translation matrices $U_{[k+,l]}^{(\alpha,\beta)}, U_{[k,l+]}^{(\alpha,\beta)}$: for instance

$$(3.14) \quad \begin{aligned} U_{[k,l+]}^{(\alpha,\beta)} &= V_{[k,l]}^{(\alpha,\beta+)} (W_{[k,(l+1)-]}^{(\alpha,\beta+)})^{-1} = (W_{[k,(l+1)-]}^{(\alpha,(\beta-1)+)})^{-1} V_{[k,l+]}^{(\alpha,(\beta-1)+)} , \\ U_{[k+,l]}^{(\alpha,\beta)} &= V_{[k,l]}^{(\alpha+, \beta)} (W_{[(k+1)-,l]}^{(\alpha+, \beta)})^{-1} = (W_{[(k+1)-,l]}^{((\alpha-1)+, \beta)})^{-1} V_{[k+,l]}^{((\alpha-1)+, \beta)} . \end{aligned}$$

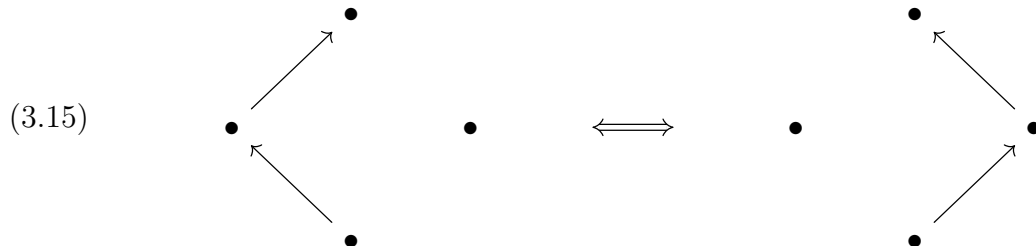
Such factorizations are well known in the theory of integrable systems, see for instance Adler[1], Sklyanin[15].

We will argue that all identities for the connection matrices are the result of those in (3.14).

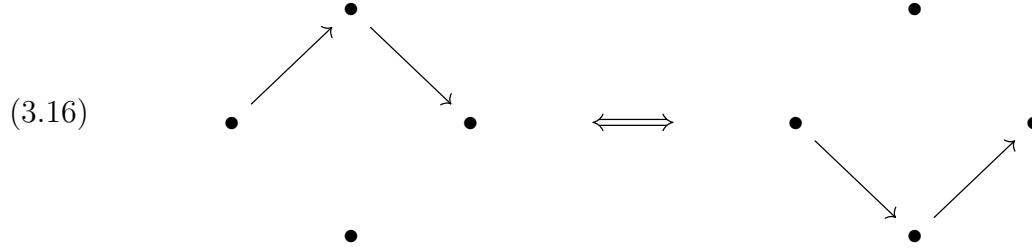
Lemma 3.4. *Any connection matrix $\Gamma_{[k,l](\alpha,\beta)}^{[k',l'](\alpha',\beta')}$ is a product of the four types of elementary matrices (3.12).*

Proof. We need to show that you can move from $\Psi^{[k,l](\alpha,\beta)}$ to $\Psi^{[k',l'](\alpha',\beta')}$ just using the elementary connections matrices. First of all the translation matrices (3.13) are products of elementary connection matrices: see (3.14). Now we can move from $\Psi^{[k,l](\alpha,\beta)}$ to $\Psi^{[k',l'](\alpha,\beta)}$ using just $U_{[(k_i)+,l_i]}$ and/or $U_{[k_i,(l_i)+]}$, keeping (α, β) fixed. Then we use the $V^{(\alpha+, \beta)}$ and/or $V^{(\alpha, \beta+)}$ to adjust the (α, β) to (α', β') to reach $\Psi^{[k',l'](\alpha',\beta')}$. \square

The expression in the Lemma of $\Gamma = \Gamma_{[k,l](\alpha,\beta)}^{[k',l'](\alpha',\beta')}$ as a product of elementary connection matrices is of course not unique. Each path from $\Psi^{[k,l](\alpha,\beta)}$ to $\Psi^{[k',l'](\alpha',\beta')}$ in the lattice of Baker functions with diagonal or antidiagonal steps gives a product expression for Γ : the diagonal steps give a factor V and the anti-diagonal steps give W factors. Now it should be clear that any two paths from $\Psi^{[k,l](\alpha,\beta)}$ to $\Psi^{[k',l'](\alpha',\beta')}$ can be deformed into each other by moves



or



The moves (3.16) correspond to identities (3.14) and moves (3.15) correspond to similar equations of the form $VW = WV$ (without inverses on W). These last equations will be equivalent to those in (3.14). So the upshot is that all equations obtained by writing an arbitrary connection matrix Γ as a product of elementary connection matrices follow from (3.14).

We will therefore concentrate on (3.14). In particular we will see that these equations will give equations for our τ functions.

We first check that our elementary connection matrices and translation matrices do not contain any negative powers of z , which is not obvious from the definitions (3.12) and (3.13).

Lemma 3.5. *For the elementary connection matrices we have the positive expressions*

$$\begin{aligned}
 V_{[k,l]}^{(\alpha, \beta)} &= g_{0+}^{[k,l](\alpha, \beta)} Q_0^{-1} (g_{0+}^{[k,l](\alpha+1, \beta)})^{-1}, \\
 V_{[k,l]}^{(\alpha, \beta+)} &= g_{0+}^{[k,l](\alpha, \beta)} Q_1^{-1} (g_{0+}^{[k,l](\alpha, \beta+1)})^{-1}, \\
 W_{[k-, l]}^{(\alpha, \beta)} &= g_{0+}^{[k,l](\alpha, \beta)} Q_0^{-1} (g_{0+}^{[k-1, l](\alpha+1, \beta)})^{-1}, \\
 W_{[k, l-]}^{(\alpha, \beta+)} &= g_{0+}^{[k,l](\alpha, \beta)} Q_1^{-1} (g_{0+}^{[k, l-1](\alpha, \beta+1)})^{-1}
 \end{aligned}
 \tag{3.17}$$

Similarly, for the translation matrices

$$\begin{aligned}
 U_{k+, l}^{(\alpha, \beta)} &= g_{0+}^{[k+1, l](\alpha, \beta)} (g_{0+}^{[k, l](\alpha, \beta)})^{-1} \\
 U_{k, l+}^{(\alpha, \beta)} &= g_{0+}^{[k, l+1](\alpha, \beta)} (g_{0+}^{[k, l](\alpha, \beta)})^{-1}
 \end{aligned}
 \tag{3.18}$$

Proof. From (3.5) it follows that

$$\begin{aligned}
 Q_0^{-1} g_-^{[k, l](\alpha+1, \beta)} g_{0+}^{[k, l](\alpha+1, \beta)} &= g_-^{[k, l](\alpha, \beta)} g_{0+}^{[k, l](\alpha, \beta)} Q_0^{-1}, \\
 Q_1^{-1} g_-^{[k, l](\alpha, \beta+1)} g_{0+}^{[k, l](\alpha, \beta+1)} &= g_-^{[k, l](\alpha, \beta)} g_{0+}^{[k, l](\alpha, \beta)} Q_1^{-1},
 \end{aligned}$$

from which the result for the V matrices follows by rearranging factors.

Similarly, using $Q_1^{-1} T_1 = Q_0^{-1}$, $Q_2^{-1} T_2 = Q_1^{-1}$ and again (3.5) we find

$$\begin{aligned}
 Q_1^{-1} g_-^{[k-1, l](\alpha+1, \beta)} g_{0+}^{[k-1, l](\alpha+1, \beta)} &= g_-^{[k, l](\alpha, \beta)} g_{0+}^{[k, l](\alpha, \beta)} Q_0^{-1}, \\
 Q_2^{-1} g_-^{[k, l-1](\alpha, \beta+1)} g_{0+}^{[k, l-1](\alpha, \beta+1)} &= g_-^{[k, l](\alpha, \beta)} g_{0+}^{[k, l](\alpha, \beta)} Q_1^{-1},
 \end{aligned}$$

giving the result for the W matrices.

Finally the positive expression for U matrices follows from

$$\begin{aligned} g_-^{[k,l](\alpha+1,\beta)} g_{0+}^{[k,](\alpha)} &= g_-^{[k](\alpha)} g_{0+}^{[k](\alpha)} Q_0^{-1}, \\ Q_1^{-1} g_-^{[k](\alpha+1)} g_{0+}^{[k](\alpha+1)} &= g_-^{[k](\alpha)} g_{0+}^{[k+1](\alpha)} Q_1^{-1}, \end{aligned}$$

and rearranging factors. □

The above Lemma tells us that the elementary connection matrices and translation matrices are positive in z , i.e., contain only z^k for $k \geq 0$. This allows us to calculate them easily in terms of τ -functions.

First note that Theorem 3.3 allows us to expand $g_-^{[k,l](\alpha,\beta)}$ and its inverse up to order z^{-1} as (we suppress the shift (α, β))

$$(3.19) \quad g_-^{[k,\ell]} = \begin{bmatrix} 1 + \mathcal{O}(z^{-1}) & \frac{1}{zh_{\underline{k-1},\ell}} + \mathcal{O}(z^{-2}) & \frac{(-1)^k}{zh_{\underline{k-1},\ell-1}} + \mathcal{O}(z^{-2}) \\ \frac{h_{\underline{k},\ell}}{z} + \mathcal{O}(z^{-2}) & 1 + \mathcal{O}(z^{-1}) & \frac{(-1)^k}{zh_{k,\underline{\ell-1}}} + \mathcal{O}(z^{-2}) \\ \frac{(-1)^{k+1}h_{\underline{k},\ell}}{z} + \mathcal{O}(z^{-2}) & \frac{(-1)^k h_{k,\underline{\ell}}}{z} + \mathcal{O}(z^{-2}) & 1 + \mathcal{O}(z^{-1}) \end{bmatrix},$$

where $\mathcal{O}(z^i)$ are terms with power of z equal to i or lower, and we define quotients of τ functions as

$$h_{\underline{k},l} = \frac{\tau_{k+1,l}}{\tau_{k,l}}, \quad h_{k,\underline{l}} = \frac{\tau_{k,l+1}}{\tau_{k,l}}, \quad h_{\underline{k},\underline{l}} = \frac{\tau_{k+1,l+1}}{\tau_{k,l}}.$$

This formula then gives the following formula for $(g_-^{[k,\ell]})^{-1}$

$$(3.20) \quad (g_-^{[k,\ell]})^{-1} = \begin{bmatrix} 1 + \mathcal{O}(z^{-1}) & -\frac{1}{zh_{\underline{k-1},\ell}} + \mathcal{O}(z^{-2}) & \frac{(-1)^{k+1}}{zh_{\underline{k-1},\ell-1}} + \mathcal{O}(z^{-2}) \\ \frac{-h_{\underline{k},\ell}}{z} + \mathcal{O}(z^{-2}) & 1 + \mathcal{O}(z^{-1}) & \frac{(-1)^{k+1}}{zh_{k,\underline{\ell-1}}} + \mathcal{O}(z^{-2}) \\ \frac{(-1)^k h_{\underline{k},\ell}}{z} + \mathcal{O}(z^{-2}) & \frac{(-1)^{k+1} h_{k,\underline{\ell}}}{z} + \mathcal{O}(z^{-2}) & 1 + \mathcal{O}(z^{-1}) \end{bmatrix}.$$

3.5. Explicit Formulae for Connection Matrices.

Lemma 3.6.

(3.21)

$$\begin{aligned}
V_{[k,l]}^{(\alpha_+, \beta)} &= \begin{bmatrix} z + A & \frac{1}{h_{\underline{k-1}, \underline{\ell}}^{(\alpha+1, \beta)}} & \frac{(-1)^k}{h_{\underline{k-1}, \underline{\ell-1}}^{(\alpha+1, \beta)}} \\ -h_{\underline{k}, l}^{(\alpha, \beta)} & 1 & 0 \\ (-1)^k h_{\underline{k}, \underline{l}}^{(\alpha, \beta)} & 0 & 1 \end{bmatrix}, \quad A = -\frac{h_{\underline{k}, \underline{\ell}}^{(\alpha, \beta)}}{h_{\underline{k-1}, \underline{\ell}}^{(\alpha+1, \beta)}} + \frac{h_{\underline{k}, \underline{\ell}}^{(\alpha, \beta)}}{h_{\underline{k-1}, \underline{\ell-1}}^{(\alpha+1, \beta)}}, \\
V_{[k,l]}^{(\alpha, \beta_+)} &= \begin{bmatrix} 1 & \frac{-1}{h_{\underline{k-1}, l}^{(\alpha, \beta)}} & 0 \\ h_{\underline{k}, l}^{(\alpha, \beta+1)} & z + B & \frac{(-1)^k}{h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta+1)}} \\ 0 & (-1)^{k+1} h_{\underline{k}, \underline{l}}^{(\alpha, \beta)} & 1 \end{bmatrix}, \quad B = \frac{-h_{\underline{k}, \underline{l}}^{(\alpha, \beta)}}{h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta+1)}} + \frac{-h_{\underline{k}, l}^{(\alpha, \beta+1)}}{h_{\underline{k-1}, l}^{(\alpha, \beta)}}, \\
W_{[k-, l]}^{(\alpha_+, \beta)} &= \begin{bmatrix} 1 & \frac{-1}{h_{\underline{k-1}, l}^{(\alpha, \beta)}} & 0 \\ h_{\underline{k-1}, l}^{(\alpha+1, \beta)} & z + C & \frac{(-1)^{k-1}}{h_{\underline{k-1}, \underline{l-1}}^{(\alpha+1, \beta)}} \\ 0 & (-1)^{k+1} h_{\underline{k}, \underline{l}}^{(\alpha, \beta)} & 1 \end{bmatrix}, \quad C = \frac{h_{\underline{k}, \underline{l}}^{(\alpha, \beta)}}{h_{\underline{k-1}, \underline{l-1}}^{(\alpha+1, \beta)}} + \frac{-h_{\underline{k-1}, l}^{(\alpha+1, \beta)}}{h_{\underline{k-1}, l}^{(\alpha, \beta)}}, \\
W_{[k, l-]}^{(\alpha, \beta_+)} &= \begin{bmatrix} 1 & 0 & \frac{(-1)^{k+1}}{h_{\underline{k-1}, \underline{l-1}}^{(\alpha, \beta)}} \\ 0 & 1 & \frac{(-1)^{k+1}}{h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta)}} \\ (-1)^{k+1} h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta+1)} & (-1)^k h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta+1)} & z + D \end{bmatrix}, \quad D = \frac{h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta+1)}}{h_{\underline{k-1}, \underline{l-1}}^{(\alpha, \beta)}} - \frac{h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta+1)}}{h_{\underline{k}, \underline{l-1}}^{(\alpha, \beta)}}.
\end{aligned}$$

Proof. For the off-diagonal entries use the expressions (3.19) and (3.20) for g_- and $(g_-)^{-1}$ in the definitions (3.17) of the elementary connection matrices, and then use positivity Lemma 3.5 to discard all terms with z^{-1} or lower. For the diagonal terms use that the determinant of all elementary connection matrices is z . \square

Lemma 3.7. *The inverses of the W matrices are*

$$(3.22) \quad (W_{[k-,l]}^{(\alpha+,\beta)})^{-1} = \frac{1}{z} \begin{bmatrix} z - \frac{h_{\underline{k-1},l}^{(\alpha+1,\beta)}}{h_{\underline{k-1},l}^{(\alpha,\beta)}} & \frac{1}{h_{\underline{k-1},l}^{(\alpha,\beta)}} & \frac{(-1)^k}{h_{\underline{k-1},l}^{(\alpha,\beta)} h_{\underline{k-1},l-1}^{(\alpha+1,\beta)}} \\ -h_{\underline{k-1},l}^{(\alpha+1,\beta)} & 1 & \frac{(-1)^k}{h_{\underline{k-1},l-1}^{(\alpha+1,\beta)}} \\ (-1)^{k+1} h_{\underline{k-1},l}^{(\alpha+1,\beta)} h_{\underline{k},\underline{l}}^{(\alpha,\beta)} & (-1)^k h_{\underline{k},\underline{l}}^{(\alpha,\beta)} & z + \frac{h_{\underline{k},\underline{l}}^{(\alpha,\beta)}}{h_{\underline{k-1},l-1}^{(\alpha+1,\beta)}} \end{bmatrix},$$

$$(3.23) \quad (W_{[k,l-]}^{(\alpha,\beta+)})^{-1} = \frac{1}{z} \begin{bmatrix} z + \frac{h_{\underline{k},l-1}^{(\alpha,\beta+1)}}{h_{\underline{k},l-1}^{(\alpha,\beta)}} & -\frac{h_{\underline{k},l-1}^{(\alpha,\beta+1)}}{h_{\underline{k-1},l-1}^{(\alpha,\beta)}} & \frac{(-1)^k}{h_{\underline{k-1},l-1}^{(\alpha,\beta)}} \\ \frac{h_{\underline{k},l-1}^{(\alpha,\beta+1)}}{h_{\underline{k},l-1}^{(\alpha,\beta)}} & z + \frac{h_{\underline{k},l-1}^{(\alpha,\beta+1)}}{h_{\underline{k-1},l-1}^{(\alpha,\beta)}} & \frac{(-1)^k}{h_{\underline{k},l-1}^{(\alpha,\beta)}} \\ (-1)^k h_{\underline{k},l-1}^{(\alpha,\beta+1)} & (-1)^{k+1} h_{\underline{k},l-1}^{(\alpha,\beta+1)} & 1 \end{bmatrix}.$$

For the translation matrices we similarly obtain the following expressions

$$(3.24) \quad U_{[k+,l]} = \begin{bmatrix} z + A & \frac{1}{h_{\underline{k},l}} & \frac{(-1)^{k+1}}{h_{\underline{k},l-1}} \\ -h_{\underline{k},l} & 0 & 0 \\ (-1)^k h_{\underline{k},\underline{l}} & 0 & 1 \end{bmatrix},$$

$$(3.25) \quad U_{[k,l+]} = \begin{bmatrix} 1 & \frac{-1}{h_{\underline{k-1},l}} & 0 \\ h_{\underline{k},l+1} & z + B & \frac{(-1)^{k+1}}{h_{\underline{k},\underline{l}}} \\ 0 & (-1)^{k+1} h_{\underline{k},\underline{l}} & 0 \end{bmatrix}.$$

Here we suppress the shifts (α, β) , and we postpone the calculation of the diagonal terms A, B .

3.6. Difference Equations from Factorizations.

Theorem 3.8. *The τ -functions satisfy the following four systems of equations*

$$\begin{aligned} ([2]) \quad & \tau_{k,l-1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha+1,\beta)} + \tau_{k+1,l}^{(\alpha,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} - \tau_{k,l-1}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)} = 0, \\ ([1]) \quad & \tau_{k+1,l+1}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)} - \tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha,\beta+1)} + \tau_{k+1,l}^{(\alpha,\beta+1)} \tau_{k,l+1}^{(\alpha,\beta)} = 0, \\ ([0]) \quad & (\tau_{k,l}^{(\alpha,\beta)})^2 = \tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha-1,\beta)} + \tau_{k+1,l+1}^{(\alpha-1,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} - \tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k-1,l}^{(\alpha+1,\beta)}, \\ ([1,1]) \quad & (\tau_{k,l}^{(\alpha,\beta)})^2 = \tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta-1)} - \tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l-1}^{(\alpha,\beta+1)} - \tau_{k-1,l}^{(\alpha,\beta-1)} \tau_{k+1,l}^{(\alpha,\beta+1)}. \end{aligned}$$

Proof. We substitute in the two factorization (2.13) the results of Lemma 3.6 and Lemma 3.7. We find rather complicated rational expressions in the τ -functions. For $U_{k+,l}^{(\alpha,\beta)}$ we get

non trivial equation in 3 components:

$$([0,0]) \quad -\frac{\tau_{k+2,l+1}^{(\alpha-1,\beta)} \tau_{k,l-1}^{(\alpha,\beta)}}{\tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k+1,l}^{(\alpha,\beta)}} + \frac{\tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)}}{\tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)}} - \frac{\tau_{k+1,l}^{(\alpha,\beta)} \tau_{k-1,l}^{(\alpha+1,\beta)}}{\tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)}} +$$

$$+ \frac{\tau_{k+1,l}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha-1,\beta)}}{\tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha,\beta)}} + \frac{\tau_{k+2,l}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha,\beta)}}{\tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k+1,l}^{(\alpha,\beta)}} - \frac{\tau_{k+1,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)}}{\tau_{k+1,l}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha+1,\beta)}} = 0$$

$$([0,2]) \quad \frac{\tau_{k,l-1}^{(\alpha,\beta)}}{\tau_{k+1,l}^{(\alpha,\beta)}} + \frac{\tau_{k-1,l-1}^{(\alpha+1,\beta)}}{\tau_{k,l}^{(\alpha+1,\beta)}} - \frac{\tau_{k,l-1}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)}}{\tau_{k+1,l}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha+1,\beta)}} = 0$$

$$([2,0]) \quad \frac{\tau_{k+2,l+1}^{(\alpha-1,\beta)}}{\tau_{k+1,l}^{(\alpha-1,\beta)}} + \frac{\tau_{k+1,l+1}^{(\alpha,\beta)}}{\tau_{k,l}^{(\alpha,\beta)}} - \frac{\tau_{k+1,l+1}^{(\alpha-1,\beta)} \tau_{k+1,l}^{(\alpha,\beta)}}{\tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha,\beta)}} = 0$$

For $U_{k,l+}^{(\alpha,\beta)}$ we get

$$([1,0]) \quad \frac{\tau_{k+1,l+1}^{a,\beta+1} \tau_{k,l}^{(\alpha,\beta)}}{\tau_{k,l+1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha,\beta+1)}} - \frac{\tau_{k+1,l+1}^{(\alpha,\beta)}}{\tau_{k,l+1}^{(\alpha,\beta)}} + \frac{\tau_{k+1,l}^{(\alpha,\beta+1)}}{\tau_{k,l}^{(\alpha,\beta+1)}} = 0$$

$$([0,1]) \quad \frac{\tau_{k-1,l}^{a,\beta-1} \tau_{k,l+1}^{(\alpha,\beta)}}{\tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l}^{(\alpha,\beta)}} + \frac{\tau_{k-1,l+1}^{(\alpha,\beta-1)}}{\tau_{k,l+1}^{(\alpha,\beta-1)}} - \frac{\tau_{k-1,l}^{(\alpha,\beta)}}{\tau_{k,l}^{(\alpha,\beta)}} = 0$$

$$([1,1]) \quad \frac{\tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k-1,l+1}^{(\alpha,\beta-1)}}{\tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l+1}^{(\alpha,\beta)}} - \frac{\tau_{k+1,l}^{(\alpha,\beta+1)} \tau_{k-1,l}^{(\alpha,\beta)}}{\tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)}} - \frac{\tau_{k,l+1}^{(\alpha,\beta)} \tau_{k,l-1}^{(\alpha,\beta+1)}}{\tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)}} +$$

$$+ \frac{\tau_{k,l+1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha,\beta-1)}}{\tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l}^{(\alpha,\beta)}} + \frac{\tau_{k,l+2}^{(\alpha,\beta-1)} \tau_{k,l}^{(\alpha,\beta)}}{\tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l+1}^{(\alpha,\beta)}} - \frac{\tau_{k,l+1}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)}}{\tau_{k,l+1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha,\beta+1)}} = 0$$

The two equations $[0,2]$ and $[2,0]$ can be brought under common denominator, and then the numerator is forced to vanish. (We think here of the coefficients c_k, d_k, e_k as formal variables, so the denominators do not vanish.) The resulting equations are the same, up to a shift in the variables. This gives the equation $[2]$ in the theorem. In the same way $[1]$ follows from $[0,1], [1,0]$.

Next we bring $[0,0]$ under common denominator. The vanishing of the numerator gives

$$(3.26) \quad \tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k+1,l}^{(\alpha,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} - \tau_{k+2,l+1}^{(\alpha-1,\beta)} \tau_{k,l-1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)} +$$

$$+ \left(\tau_{k+2,l}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha+1,\beta)} - \tau_{k+1,l}^{(\alpha+1,\beta)} \tau_{k+1,l}^{(\alpha-1,\beta)} \right) (\tau_{k,l}^{(\alpha,\beta)})^2 -$$

$$\left(\tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k-1,l}^{(\alpha+1,\beta)} - \tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha-1,\beta)} \right) (\tau_{k+1,l}^{(\alpha,\beta)})^2 = 0$$

In the first term we substitute

$$\tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k+1,l}^{(\alpha-1,\beta)} = \tau_{k+1,l}^{(\alpha,\beta)} \tau_{k+1,l+1}^{(\alpha-1,\beta)} - \tau_{k+2,l+1}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha,\beta)},$$

which follows from $[2]$ by change of variables $k \mapsto k+1, l \mapsto l+1, \alpha+1 \mapsto \alpha$. The first term then becomes

$$\tau_{k+1,l}^{(\alpha,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} \left(\tau_{k+1,l}^{(\alpha,\beta)} \tau_{k+1,l+1}^{(\alpha-1,\beta)} - \tau_{k+2,l+1}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha,\beta)} \right).$$

In the second term of (3.26) we use [2] in the form

$$\tau_{k,l-1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha+1,\beta)} = \tau_{k,l-1}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)} - \tau_{k+1,l}^{(\alpha,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)}$$

so that the second term becomes

$$-\tau_{k+2,l+1}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha,\beta)} \left(\tau_{k,l-1}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha,\beta)} - \tau_{k+1,l}^{(\alpha,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} \right).$$

After cancellation of two terms and collecting like terms we find for (3.26)

$$(3.27) \quad (\tau_{k,l}^{(\alpha,\beta)})^2 \left(\tau_{k+2,l}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha+1,\beta)} - \tau_{k+1,l}^{(\alpha+1,\beta)} \tau_{k+1,l}^{(\alpha-1,\beta)} - \tau_{k+2,l+1}^{(\alpha-1,\beta)} \tau_{k,l-1}^{(\alpha+1,\beta)} \right) =$$

$$(\tau_{k+1,l}^{(\alpha,\beta)})^2 \left(\tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k-1,l}^{(\alpha+1,\beta)} - \tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha-1,\beta)} - \tau_{k+1,l+1}^{(\alpha-1,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} \right).$$

Now observe that the square factor on the right in (3.27) is obtained from the square factor on the left by a shift $k \mapsto k+1$, and similarly, mutatis mutandis, for the terms in the big parentheses. Therefore, if we know that

$$(3.28) \quad (\tau_{k,l}^{(\alpha,\beta)})^2 = \tau_{k,l}^{(\alpha+1,\beta)} \tau_{k,l}^{(\alpha-1,\beta)} + \tau_{k+1,l+1}^{(\alpha-1,\beta)} \tau_{k-1,l-1}^{(\alpha+1,\beta)} - \tau_{k+1,l}^{(\alpha-1,\beta)} \tau_{k-1,l}^{(\alpha+1,\beta)},$$

then (3.27) tells us that also

$$(\tau_{k+1,l}^{(\alpha,\beta)})^2 = \tau_{k+1,l}^{(\alpha+1,\beta)} \tau_{k+1,l}^{(\alpha-1,\beta)} + \tau_{k+2,l+1}^{(\alpha-1,\beta)} \tau_{k,l-1}^{(\alpha+1,\beta)} - \tau_{k+2,l}^{(\alpha-1,\beta)} \tau_{k,l}^{(\alpha+1,\beta)}.$$

So it suffice to check (3.28) for $k=0$, which reads

$$(\tau_{0,l}^{(\alpha,\beta)})^2 = \tau_{0,l}^{(\alpha+1,\beta)} \tau_{0,l}^{(\alpha-1,\beta)}.$$

But this last equation is true by Example 4 of section 3.2: the τ -function $\tau_{0,l}^{(\alpha,\beta)}$ is independent of α , see also (3.3). This proves part [0] of the theorem.

Finally we use the [1,1] component of the two expressions of $U_{k,l+}^{(\alpha,\beta)}$. Bringing it under common denominator, the vanishing of the numerator gives

$$(3.29) \quad \tau_{k+1,l}^{(\alpha,\beta+1)} \tau_{k-1,l}^{(\alpha,\beta)} \tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l+1}^{(\alpha,\beta)} - \tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k-1,l+1}^{(\alpha,\beta-1)} \tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)} +$$

$$\left(\tau_{k,l+1}^{(\alpha,\beta+1)} \tau_{k,l+1}^{(\alpha,\beta-1)} - \tau_{k,l+2}^{(\alpha,\beta-1)} \tau_{k,l}^{(\alpha,\beta+1)} \right) (\tau_{k,l}^{(\alpha,\beta)})^2 +$$

$$\left(\tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l-1}^{(\alpha,\beta+1)} - \tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta-1)} \right) (\tau_{k,l+1}^{(\alpha,\beta)})^2 = 0$$

In the first term substitute

$$\tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k-1,l}^{(\alpha,\beta)} = \tau_{k,l+1}^{(\alpha,\beta)} \tau_{k-1,l}^{(\alpha,\beta-1)} + \tau_{k,l}^{(\alpha,\beta)} \tau_{k-1,l+1}^{(\alpha,\beta-1)},$$

which is obtained from [1] by shifts $k \mapsto k-1, \beta \mapsto \beta-1$. The first term becomes

$$\tau_{k+1,l}^{(\alpha,\beta+1)} \tau_{k,l+1}^{(\alpha,\beta)} \left(\tau_{k,l+1}^{(\alpha,\beta)} \tau_{k-1,l}^{(\alpha,\beta-1)} + \tau_{k,l}^{(\alpha,\beta)} \tau_{k-1,l+1}^{(\alpha,\beta-1)} \right).$$

In the second term we also use [1], in the form

$$\tau_{k+1,l+1}^{(\alpha,\beta)} \tau_{k,l}^{(\alpha,\beta+1)} = \tau_{k+1,l+1}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)} + \tau_{k+1,l}^{(\alpha,\beta+1)} \tau_{k,l+1}^{(\alpha,\beta)},$$

transforming the second term to

$$-\tau_{k-1,l+1}^{(\alpha,\beta-1)} \tau_{k,l}^{(\alpha,\beta)} \left(\tau_{k+1,l+1}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta)} + \tau_{k+1,l}^{(\alpha,\beta+1)} \tau_{k,l+1}^{(\alpha,\beta)} \right).$$

After cancellation of two terms and collecting like terms (3.29) becomes

$$(3.30) \quad \left(\tau_{k,l+1}^{(\alpha,\beta+1)} \tau_{k,l+1}^{(\alpha,\beta-1)} - \tau_{k,l+2}^{(\alpha,\beta-1)} \tau_{k,l}^{(\alpha,\beta+1)} - \tau_{k-1,l+1}^{(\alpha,\beta-1)} \tau_{k+1,l+1}^{(\alpha,\beta+1)} \right) (\tau_{k,l}^{(\alpha,\beta)})^2 = \\ \left(\tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta-1)} - \tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l-1}^{(\alpha,\beta+1)} - \tau_{k-1,l}^{(\alpha,\beta-1)} \tau_{k+1,l}^{(\alpha,\beta+1)} \right) (\tau_{k,l+1}^{(\alpha,\beta)})^2.$$

As before, (3.30) implies that if

$$(3.31) \quad (\tau_{k,l}^{(\alpha,\beta)})^2 = \tau_{k,l}^{(\alpha,\beta+1)} \tau_{k,l}^{(\alpha,\beta-1)} - \tau_{k,l+1}^{(\alpha,\beta-1)} \tau_{k,l-1}^{(\alpha,\beta+1)} - \tau_{k-1,l}^{(\alpha,\beta-1)} \tau_{k+1,l}^{(\alpha,\beta+1)},$$

the same equation holds after a shift $l \mapsto l+1$. So we reduce to the case of $l=0$ of (3.31):

$$(\tau_{k,0}^{(\alpha,\beta)})^2 = \tau_{k,0}^{(\alpha,\beta+1)} \tau_{k,0}^{(\alpha,\beta-1)} - \tau_{k-1,0}^{(\alpha,\beta-1)} \tau_{k+1,0}^{(\alpha,\beta+1)},$$

using Item 1 of the examples in subsection 3.2. Now by Item 3 of these examples $\tau_{k,0}^{(\alpha,\beta)} = \tau_k^{(\alpha-\beta)}$. So the case $l=0$ is just the Q -system of Theorem 2.5. This proves the final part [1, 1] of the theorem. \square

APPENDIX A. MULTI COMPONENT FERMIONS AND SEMI INFINITE WEDGE SPACE

A.1. Introduction. In the main text we will work with $n \times n$ matrices (depending on a spectral parameter z) for $n=2$ or 3 . In this appendix we will not specify n , as the theory of n -component fermions and associated semi-infinite wedge space does not significantly depend on n . A convenient reference for background and more details is ten Kroode and van der Leur [16].

A.2. Semi Infinite Wedge Space. Let $\{e_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\}$ denote

the standard basis of \mathbb{C}^n . Denote the corresponding elementary matrices by E_{ab} (such that $E_{ab}e_c = \delta_{bc}e_a$); they are also indexed by integers $0, 1, \dots, n-1$. We will also need the loop space of \mathbb{C}^n , denoted by

$$(A.1) \quad H^{(n)} = \mathbb{C}^n \otimes \mathbb{C}[z, z^{-1}],$$

with basis $e_a^k = e_a z^k$, for $a=0, \dots, n-1$ and $k \in \mathbb{Z}$. Let $F^{(n)}$ be the n -component fermionic Fock space, the semi infinite wedge space based on $H^{(n)}$. It is spanned by semi-infinite wedges

$$\omega = w_0 \wedge w_1 \wedge w_3 \wedge \dots, \quad w_i \in H^{(n)},$$

where the w_i satisfy some restrictions that we will presently discuss. Semi-infinite wedges obey the usual rules of exterior algebra, like multilinearity in each factor and antisymmetry under exchange of two factors.

To formulate the restrictions on the w_i that can appear in the wedge ω above we introduce the Clifford algebra $Cl^{(n)}$ acting on $F^{(n)}$: it is generated by exterior and interior products with $e_a^k = e_a z^k$, denoted by $e(e_a^k)$ and $i(e_a^k)$, defined as wedging and unwedging:

$$e(e_a^k)\alpha = e_a^k \wedge \alpha, \quad i(e_a^k)\alpha = \beta, \quad \text{if } \alpha = e_a^k \wedge \beta.$$

It is useful to collect the generators of the Clifford algebra in generating series. Therefore, define *fermion fields*

$$\psi_a^\pm(w) = \sum_{k \in \mathbb{Z}} {}_a\psi_{(k)}^\pm w^{-k-1}, \quad a = 0, 1, \dots, n-1$$

where

$$(A.2) \quad {}_a\psi_{(k)}^+ = e(e_a^k) = e_a z^k \wedge, \quad {}_a\psi_{(k)}^- = i(e_a z^{-k-1}).$$

The fermionic fields satisfy commutation relations

$$[\psi_a^\pm(z), \psi_b^\pm(w)]_+ = 0, \quad [\psi_a^+(w_1), \psi_b^-(w_2)]_+ = \delta_{ab} \delta(w_1, w_2),$$

where the formal delta distribution is defined by

$$(A.3) \quad \delta(z, w) = \sum_{k \in \mathbb{Z}} z^k w^{-k-1}.$$

Let v_0 be the vacuum vector

$$(A.4) \quad v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \wedge \begin{bmatrix} z \\ 0 \\ 0 \\ \vdots \end{bmatrix} \wedge \begin{bmatrix} 0 \\ z \\ 0 \\ \vdots \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z \end{bmatrix} \wedge \begin{bmatrix} z^2 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \wedge \begin{bmatrix} 0 \\ z^2 \\ 0 \\ \vdots \end{bmatrix} \wedge \cdots$$

Then we define $F^{(n)}$ to be span of the wedges obtained by acting on the vacuum v_0 by monomials in the wedging and unwedging operators. To get a basis for $F^{(n)}$ we specify an ordering on the wedging/unwedging operators acting on $F^{(n)}$.

Definition A.1. An elementary wedge in $F^{(n)}$ is an element $\omega = Mv_0$, where

$$M = M_{n-1} \dots M_1 M_0, \quad M_a = M_a^+ M_a^-, \quad a = n-1, n-2, \dots, 2, 1$$

where

$$M_a^\pm = {}_a\psi_{(k_1)}^\pm {}_a\psi_{(k_2)}^\pm \cdots {}_a\psi_{(k_s)}^\pm, \quad k_1 < k_2 < \cdots < k_s \leq 1,$$

is a monomial in ${}_a\psi_{(k)}^\pm$ for $k \leq -1$, ordered in increasing order from left to right.

The statement that the elementary wedges form a basis for $F^{(n)}$ follows from the Poincaré-Birkhoff-Witt theorem for the Lie superalgebra underlying the Clifford algebra.

We define a bilinear form, denoted $\langle \cdot, \cdot \rangle$, on $F^{(n)}$ by declaring the elementary wedges to be orthonormal. We have then

$$(A.5) \quad \langle {}_a\psi_{(k)}^+ v, w \rangle = \langle v, {}_a\psi_{(-k-1)}^- w \rangle.$$

and

$$(A.6) \quad \langle \psi_a^+(z) v, w \rangle = \langle v, \psi_a^-(z^{-1}) z^{-1} w \rangle.$$

The n -component fermionic Fock space $F^{(n)}$ has a grading by the Abelian group \mathbb{Z}^n , i.e., we have a decomposition $F^{(n)} = \bigoplus_{\alpha \in \mathbb{Z}^n} F_\alpha^{(n)}$. The vacuum has degree $\alpha = 0$. To describe the grading introduce a basis in \mathbb{Z}^n by

$$\delta_a = (0 \ 0 \ \dots \ 1 \ \dots \ 0), \quad \text{where the 1 is in position } a = 0, 1, \dots, n-1$$

The grading on $F^{(n)}$ induces a grading on linear maps on $F^{(n)}$: if $L: F^{(n)} \rightarrow F^{(n)}$ has the property that there exist a $\delta \in \mathbb{Z}^n$ so that for all $\omega \in \mathbb{Z}^n$ L restricts to a map $F_\omega^{(n)} \rightarrow F_{\omega+\delta}$, then we say that L has degree δ . Then the grading is uniquely determined by declaring

wedging operators $e_a z^k \wedge$ to have degree δ_a , and the unwedging operators $i(e_a z^k)$ to have degree $-\delta_a$. The fields $\psi_a^\pm(z)$ have degree $\pm\delta_a$.

The *total degree* of an element ω of degree α is just the sum of the entries in the degree row vector α .

A.3. Fermionic Translation Operators and Translation Group. Besides the action of fermion operators ${}_a\psi_{(k)}^\pm$ we also have on $F^{(n)}$ the action of fermionic translation operators $Q_a: F^{(n)} \rightarrow F^{(n)}$, $a = 0, 1, \dots, n-1$, given by

$$(A.7) \quad Q_a v_0 = \psi_a^+(z) v_0 \big|_{z=0} = {}_a\psi_{(-1)}^+ v_0,$$

and

$$(A.8) \quad \psi_a^\pm(z) Q_a = z^{\pm 1} Q_a \psi_a^{\pm 1}(z),$$

$$(A.9) \quad \psi_a^\pm(z) Q_b = -Q_b \psi_a^{\pm 1}(z), \quad a \neq b,$$

$$(A.10) \quad Q_a Q_b = -Q_b Q_a, \quad a \neq b.$$

The Q_a are invertible. The $Q_a^{\pm 1}$ have degree $\pm\delta_a$.

The Q_a are unitary for the standard bilinear form of $F^{(n)}$:

$$(A.11) \quad \langle Q_a v, w \rangle = \langle v, Q_a^{-1} w \rangle, \quad a = 0, 1, \dots, n-1.$$

The fermionic translation operators belong to the central extension of the loop group $\widetilde{\text{GL}}_n$ (acting on $F^{(n)}$). They are lifts of commuting elements of the non extended loop group, see for instance [2], Proposition 5.3.4. We indicate these elements of the loop group also by Q_a , and we have

$$(A.12) \quad Q_a = \sum_{b \neq a} E_{bb} + z^{-1} E_{aa}.$$

The group generated by $Q_a, a = 0, 1, \dots, n-1$, contains a subgroup of elements of total degree zero, generated by the translation operators $T_s = Q_s Q_{s-1}^{-1}$, $s = 1, 2, \dots, n$, of degree $\delta_s - \delta_{s-1}$. Another set of generators for this subgroup is also useful: define

$$T_{ab} = Q_a Q_b^{-1},$$

of degree $\delta_a - \delta_b$.

Lemma A.2. (1) $T_{ab} Q_c = Q_c T_{ab}$ if $c \neq a, b$.

(2) For $a \neq b$

$$(Q_a Q_b^{-1})^m = (-1)^{\frac{m(m-1)}{2}} Q_a^m Q_b^{-m}.$$

(3) For all $k, l \in \mathbb{Z}$ we have

$$T_2^k T_1^l = (-1)^{\frac{k(k-1)}{2} + \frac{l(l-1)}{2}} Q_2^k Q_1^{l-k} Q_0^{-l}.$$

(4)

$$T_{10}^\alpha T_{20}^\beta T_{21}^\gamma = (-1)^{\frac{\alpha(\alpha-1)}{2} + \frac{\beta(\beta-1)}{2} + \frac{\gamma(\gamma-1)}{2} + \alpha\gamma} Q_2^{\alpha+\beta} Q_1^{\alpha-\gamma} Q_0^{-\alpha-\beta}.$$

(5)

$$T_{10}^\alpha T_{20}^\beta T_{21}^\gamma = (-1)^{\frac{\beta(\beta-1)}{2} + \alpha\beta + \alpha\gamma + \beta\gamma} T_2^{\beta+\gamma} T_1^{\alpha+\beta}$$

Proof. Part (1) is clear. Part (2) is a simple induction. For part (3) we get

$$T_2^k T_1^l = (Q_2 Q_1^{-1})^k (Q_1 Q_0^{-1})^l = (-1)^{\frac{k(k-1)}{2}} (-1)^{\frac{l(l-1)}{2}} Q_2^k Q_1^{-k} Q_1^l Q_0^{-l}.$$

Similarly for part (4) we have

$$\begin{aligned} T_{10}^\alpha T_{20}^\beta T_{21}^\gamma &= (-1)^{\frac{\alpha(\alpha-1)}{2} + \frac{\beta(\beta-1)}{2} + \frac{\gamma(\gamma-1)}{2}} Q_1^\alpha Q_0^{-\alpha} Q_2^\beta Q_0^{-\beta} Q_2^\gamma Q_1^{-\gamma} = \\ &= (-1)^{\frac{\alpha(\alpha-1)}{2} + \frac{\beta(\beta-1)}{2} + \frac{\gamma(\gamma-1)}{2}} Q_2^\beta Q_1^\alpha Q_2^\gamma Q_1^{-\gamma} Q_0^{-\alpha} Q_0^{-\beta} = \\ &= (-1)^{\frac{\alpha(\alpha-1)}{2} + \frac{\beta(\beta-1)}{2} + \frac{\gamma(\gamma-1)}{2} + \alpha\gamma} Q_2^{\beta+\gamma} Q_1^{\alpha-\gamma} Q_0^{-\alpha-\beta} \end{aligned}$$

Finally, for part (5) we substitute

$$Q_2^{\beta+\gamma} Q_1^{\alpha-\gamma} Q_0^{-\alpha-\beta} = (-1)^{\frac{\alpha(\alpha-1)}{2} + \frac{\gamma(\gamma-1)}{2} + \alpha\beta + \beta\gamma} T_2^{\beta+\gamma} T_1^{\alpha+\beta},$$

(from part (3)), in the right hand side of part (4). The result then follows from

$$(-1)^{\frac{\alpha(\alpha-1)}{2} + \frac{\beta(\beta-1)}{2} + \frac{\gamma(\gamma-1)}{2} + \alpha\gamma + \frac{\alpha(\alpha-1)}{2} + \frac{\gamma(\gamma-1)}{2} + \alpha\beta + \beta\gamma} = (-1)^{\frac{\beta(\beta-1)}{2} + \alpha\beta + \alpha\gamma + \beta\gamma}.$$

□

Define the ordered product of $k \geq 0$ fermions by

$$(A.13) \quad \prod_{l=1}^{\rightarrow k} {}_a \psi_{(-l)}^\pm = {}_a \psi_{(-k)}^\pm \cdots {}_a \psi_{(-1)}^\pm.$$

The empty product is as usual the identity.

Lemma A.3. (1) For all $k \in \mathbb{Z}$, $k \neq 0$

$$Q_a^k v_0 = \begin{cases} v_0 & k = 0 \\ \prod_{l=1}^{\rightarrow k} {}_a \psi_{(-l)}^+ v_0 & k > 0, \\ \prod_{l=1}^{\rightarrow -k} {}_a \psi_{(-l)}^- v_0 & k < 0. \end{cases}$$

(2) For all $\alpha, \beta \geq 0$

$$\begin{aligned} Q_1^\beta Q_0^\alpha v_0 &= (-1)^{\beta\gamma} \prod_{l=1}^{\rightarrow \alpha} \psi_{(-l)}^+ \prod_{m=1}^{\rightarrow \beta} \psi_{(-m)}^+ v_0 = \\ &= \prod_{m=1}^{\rightarrow \beta} \psi_{(-m)}^+ \prod_{l=1}^{\rightarrow \gamma} \psi_{(-l)}^+ v_0 = \\ &= e_1 z^{-\beta} \wedge e_1 z^{1-\beta} \wedge \cdots \wedge e_1 z^{-1} \wedge e_0 z^{-\alpha} \wedge e_0 z^{1-\alpha} \wedge \cdots \wedge e_0 z^{-1} \wedge v_0. \end{aligned}$$

□

The translation operators T_s or T_{ab} are also unitary, just as the fermionic translation operators: from (A.11) it follows that

$$(A.14) \quad \langle T v, w \rangle = \langle v, T^{-1} w \rangle.$$

A.4. The Lie Algebra $\widetilde{\mathfrak{gl}}_n$ and Fermions. The loop algebra $\widetilde{\mathfrak{gl}}_n$ is defined as the Lie subalgebra of $gl(H^{(n)})$ generated by $E_{ab}z^k$, $a, b = 0, 1, \dots, n-1$, $k \in \mathbb{Z}$, where

$$(E_{ab}z^k) \cdot (e_c z^m) = \delta_{bc} e_a z^{k+m}.$$

The loop algebra $\widetilde{\mathfrak{gl}}_n$ does not quite act on $F^{(n)}$. One would like to define on $F^{(n)}$

$$E_{ab}z^k \mapsto \sum_{l \in \mathbb{Z}} (e_a z^{k+l} \wedge) (i(e_b z^l)) = \sum_{l \in \mathbb{Z}} {}_a\psi_{(k+l)}^+ {}_b\psi_{(-l-1)}^-.$$

However, considering the action of $E_{aa}z^0$ on the vacuum v_0 we would get:

$$E_{aa}v_0 = \sum_{l \geq 0} v_0,$$

(since $(e_a z^l \wedge) (i(e_a z^l)) v_0 = v_0$ for $l \geq 0$), so that these diagonal elements would have a divergent action. Therefore we introduce a normal ordering on fermion fields ([9]) by

$$: \psi_a^+(z) \psi_b^-(w) : = \psi_{a,\text{cr}}^+(z) \psi_b^-(w) - \psi_b^-(w) \psi_{a,\text{ann}}^+(z),$$

where the creation and annihilation parts of a fermion field are given by

$$\psi_{\text{cr}}(z) = \sum_{k \geq 0} \psi_{(-k-1)} z^k, \quad \psi_{\text{ann}}(z) = \sum_{k \geq 0} \psi_{(k)} z^{-k-1}.$$

In terms of the components the normal ordering means

$$: {}_a\psi_{(k)}^+ {}_b\psi_{(l)}^- : = \begin{cases} {}_a\psi_{(k)}^+ {}_b\psi_{(l)}^- & k \geq 0, \\ -{}_b\psi_{(l)}^- {}_a\psi_{(k)}^+ & k < 0. \end{cases}$$

Then we can define an action on $F^{(n)}$ by

$$E_{ab}z^k \mapsto : \sum_{l \in \mathbb{Z}} (e_a z^{k+l} \wedge) (i(e_b z^l)) : = \sum_{l \in \mathbb{Z}} : {}_a\psi_{(k+l)}^+ {}_b\psi_{(-l-1)}^- : .$$

This gives a central extension

$$(A.15) \quad 0 \rightarrow \mathbb{C} \rightarrow \widetilde{\mathfrak{gl}}_n \rightarrow \widetilde{\mathfrak{gl}}_n \rightarrow 0$$

Introduce a generating series of loop algebra elements by:

$$(A.16) \quad E_{ab}(z_1) = \sum_{l \in \mathbb{Z}} E_{ab} z_1^l z_1^{-l-1}.$$

Acting on $F^{(n)}$ it can be represented by a normal ordered product of fermion fields:

$$(A.17) \quad E_{ab}(z_1) = : \psi_a^+(z_1) \psi_b^-(z_1) : .$$

Indeed,

$$\begin{aligned} E_{ab}(z_1) &= \sum_{k, l \in \mathbb{Z}} : {}_a\psi_{(k+l)}^+ {}_b\psi_{(-l-1)}^- : z_1^{-k-1} = \\ &= \sum_{k, l \in \mathbb{Z}} : ({}_a\psi_{(k+l)}^+ z_1^{-k-l-1}) ({}_b\psi_{(-l-1)}^- z_1^l) : = : \psi_a^+(z_1) \psi_b^-(z_1) : . \end{aligned}$$

As field on $F^{(n)}$ $E_{ab}(z_1)$ has degree $\delta_a - \delta_b$.

Equation (A.17) is the reason we chose the encoding wedging and unwedging operators as coefficients of fermion fields according to (A.2).

We will need the commutator of the generating series of Lie algebra elements with fermionic translation operators.

Lemma A.4. *For all $\alpha, \beta \in \mathbb{Z}$ we have*

$$Q_0^\beta Q_1^{-\alpha} E_{1,0}(z) Q_1^\alpha Q_0^{-\beta} = (-1)^{\alpha+\beta} z^{\alpha+\beta} E_{1,0}(z).$$

A.5. Root Lattice. Recall the group \mathbb{Z}^n that gives a grading for fermionic Fock space $F^{(n)}$. It contains as a subgroup the root lattice A_{n-1} , generated by

$$\alpha_i = \delta_{i+1} - \delta_i, \quad i = 1, 2, \dots, n-1$$

So

$$A_{n-1} = \mathbb{Z}^{n-1} \alpha \subset \mathbb{Z}^n.$$

We will call elements in A_{n-1} of the form $\alpha = \sum_{i=1}^{n-1} n_i \alpha_i$ positive roots if $n_i \geq 0$.

The translation group is also graded by A_{n-1} : the generator $T_s = Q_s Q_{s-1}^{-1}$ has degree α_s . Similarly the Lie algebra generating fields $E_{ab}(z)$ have

$$\deg(E_{aa-1}(z)) = \alpha, \quad \deg(E_{01}(z)) = -\alpha.$$

APPENDIX B. EXPRESSIONS FOR THE τ -FUNCTIONS

In this appendix we prove Theorem 2.1 and Theorem 3.1. This gives expressions for the τ -functions in terms of coordinates on the lower triangular subgroup \mathcal{N} of $\widehat{\text{GL}}_n$, for $n = 2, 3$.

B.1. The case of $n = 2$, Theorem 2.1. The τ -functions for $\widehat{\text{GL}}_2$ are given as matrix elements on $F^{(2)}$:

$$(B.1) \quad \tau_k^{(\alpha)} = \langle T^k v_0, g_C^{(\alpha)} v_0 \rangle,$$

where the element $g_C^{(\alpha)}$ of the lower triangular sub group \mathcal{N} of $\widehat{\text{GL}}_2$ has projection given by (2.3). We write it as

$$g_C^{(\alpha)} = \exp(\Gamma_C^{(\alpha)}) = \sum_{l=0}^{\infty} \frac{1}{l!} (\Gamma_C^{(\alpha)})^l,$$

where

$$(B.2) \quad \Gamma_C^{(\alpha)} = \text{Res}_{z_1} (C^{(\alpha)}(z_1) E_{10}(z_1)),$$

and $C^{(\alpha)}(z)$ is given by (2.2), and the generating series of loop algebra elements $E_{10}(z_1)$ by (A.16).

Now $T^k v_0$ has degree $k(\delta_1 - \delta_0)$ in $F^{(2)}$ and $\Gamma_C^{(\alpha)}$ has degree $\delta_1 - \delta_0$, since $E_{10}(z_1)$ has. In $F^{(2)}$ homogeneous elements of different degree are orthogonal for \langle, \rangle . Hence only the $l = k$ term contributes to (B.1) and

$$\tau_k^{(\alpha)} = \frac{1}{k!} \langle T^k v_0, (\Gamma_C^{(\alpha)})^k v_0 \rangle.$$

Recall that $C^{(\alpha)}(z) = \sum_{n \in \mathbb{Z}} c_{n+\alpha} z^{-n-1}$. Associated to the series $C^{(\alpha)}(z)$ we have a \mathbb{C} -linear map

$$c^{(\alpha)}: \mathbb{C}[[z, z^{-1}]] \rightarrow B = \mathbb{C}[[c_i]]_{i \in \mathbb{Z}}, \quad f(z) \mapsto \text{Res}_z (C^{(\alpha)}(z) f(z)).$$

We will need multiple copies of the map $c^{(\alpha)}$ acting on series in variables z_1, z_2, \dots . We put $c_i^{(\alpha)}(f(z_i)) = c^{(\alpha)}(f(z))$, for $i = 1, 2, \dots$, and impose linearity in the variables z_j , $j \neq i$, i.e., the condition that (for instance) $c_i^{(\alpha)}(f(z_i) z_j^m) = z_j^m c_i^{(\alpha)}(f(z_i))$.

Then we can write

$$(\Gamma_C^{(\alpha)})^l = \prod_{i=1}^l c_i^{(\alpha)} \prod_{j=1}^l E_{10}(z_j),$$

and so

$$\begin{aligned} \tau_k^{(\alpha)} &= \frac{1}{k!} \prod_{i=1}^k c_i^{(\alpha)} \left(\langle T^k v_0, \prod_{j=1}^k E_{10}(z_j) v_0 \rangle \right) = \frac{1}{k!} \prod_{i=1}^k c_i^{(\alpha)} \left(\langle T^k v_0, \prod_{j=1}^k \psi_1^+(z_j) \psi_0^-(z_j) v_0 \rangle \right) = \\ (B.3) \quad &= \frac{1}{k!} \prod_{i=1}^k c_i^{(\alpha)} \left(\langle Q_1^k Q_0^{-k} v_0, \prod_{s=1}^k \psi_1^+(z_s) \prod_{t=1}^k \psi_0^-(z_t) v_0 \rangle \right). \end{aligned}$$

Here we use that in the expression (A.17) of $E_{10}(z_1)$ in fermions we don't need normal ordering, and also Lemma A.2, part 2.

Next we use the factorization Lemma D.1 and the Lemma E.2 to calculate the factors involving $\psi_0(z)$ and $\psi_1(z)$ separately; we find

$$\tau_k^{(\alpha)} = \frac{1}{k!} \prod_{i=1}^k c_i^{(\alpha)} \left(\det(V_{\{z_i\}}^{(k)})^2 \right).$$

Here $V_{\{z\}}^{(k)}$ is the Vandermonde matrix (D.1).

This proves the first part of Theorem 2.1, since $\det(V_{\{z_i\}}^{(k)}) = \prod_{k \geq j > i \geq 1} (z_i - z_j)$.

For the second part we need a formula for the square of a Vandermonde determinant. Let the permutation group S_k act on $\mathbb{C}[z_1, z_2, \dots, z_k]$ by permuting the subscripts.

Lemma B.1.

$$(B.4) \quad \det(V_{\{z_i\}}^{(k)})^2 = \sum_{\sigma \in S_k} \det(z_{\sigma(i)}^{i+j-2})_{i,j=1}^k.$$

Proof. The right hand side of (B.4) can be written as

$$\sum_{\sigma \in S_k} \sigma \det \begin{pmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{k-1} \\ z_2 & z_2^2 & z_2^3 & \dots & z_2^k \\ z_3^2 & z_3^3 & z_3^4 & \dots & z_3^{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k^{k-1} & z_k^k & z_k^{k+1} & \dots & z_k^{2k-2} \end{pmatrix}.$$

From this we see that the $\sigma = e$ term on the RHS of the identity we want to prove is

$$A = \det(z_i^{i+j-2}) = \prod_{s=1}^k z_s^{s-1} \det(z_i^{j-1}) = \prod_{s=1}^k z_s^{s-1} \det(V_{\{z_i\}}^{(k)}).$$

Now for every permutation σ in S_k we have

$$\sigma(A) = (-1)^{|\sigma|} \prod_{s=1}^k z_{\sigma(s)}^{s-1} \det(V_{\{z_i\}}^{(k)}).$$

Sum over all permutations

$$\sum_{\sigma} \sigma(A) = \sum_{\sigma \in S_k} (-1)^{|\sigma|} \prod_{s=1}^k z_{\sigma(s)}^{s-1} \det(V_{\{z_i\}}^{(k)}) = \det(V_{\{z_i\}}^{(k)})^2.$$

□

Introduce a monomial column vector $v(z) = \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{k-1} \end{pmatrix}$. Define the Hankel matrix (with coefficients in B)

$$\begin{aligned} T_k^{(\alpha)} &= c^{(\alpha)}(v(z)v(z)^T) = c^{(\alpha)} \begin{pmatrix} 1 & z & z^2 & \dots & z^{k-1} \\ z & z^2 & z^3 & \dots & z^k \\ z^2 & z^3 & z^4 & \dots & z^{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{k-1} & z^k & z^{k+1} & \dots & z^{2k-1} \end{pmatrix} = \\ &= \begin{pmatrix} c_\alpha & c_{\alpha+1} & c_{\alpha+2} & \dots & c_{\alpha+k-1} \\ c_{\alpha+1} & c_{\alpha+2} & c_{\alpha+3} & \dots & c_{\alpha+k} \\ c_{\alpha+2} & c_{\alpha+3} & c_{\alpha+4} & \dots & c_{\alpha+k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{\alpha+k-1} & c_{\alpha+k} & c_{\alpha+k+1} & \dots & c_{\alpha+2k-2} \end{pmatrix}, \end{aligned}$$

where we apply $c^{(\alpha)}$ component wise. Then the value of $\det(T_k^{(\alpha)}) \in B$ can be calculated in terms of Vandermonde determinants.

Lemma B.2. *If $V_{\{z\}}^{(k)}$ is the Vandermonde matrix (D.1) then*

$$\det(T_k^{(\alpha)}) = \frac{1}{k!} \prod_{i=1}^k c_i^{(\alpha)} \left(\det(V_{\{z_i\}}^{(k)})^2 \right).$$

Proof. By the trivial observation that

$$c(f(z)) \cdot c(g(z)) = c_1(f(z_1)) \cdot c_2(g(z_2))$$

we have

$$\begin{aligned} \det(T_k^{(\alpha)}) &= \prod_{i=1}^k c_i^{(\alpha)} \det \begin{pmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{k-1} \\ z_2 & z_2^2 & z_2^3 & \dots & z_2^k \\ z_3^2 & z_3^3 & z_3^4 & \dots & z_3^{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k^{k-1} & z_k^k & z_k^{k+1} & \dots & z_k^{2k-2} \end{pmatrix} = \\ &= \prod_{i=1}^k c_i^{(\alpha)} \det(z_i^{i+j-2}). \end{aligned}$$

Now for any polynomial $f(z_1, z_2, \dots, z_k)$ and any permutation $\sigma \in S_k$ we have $\prod_{i=1}^k c_i^{(\alpha)}(\sigma f(z_1, z_2, \dots, z_k)) = \prod_{i=1}^k c_i^{(\alpha)}(f(z_1, z_2, \dots, z_k))$. Hence

$$\det(T_k^{(\alpha)}) = \frac{1}{k!} \prod_{i=1}^k c_i^{(\alpha)} \left(\sum_{\sigma} \det(z_i^{i+j-2}) \right),$$

and the lemma follows from the previous Lemma B.1. □

This finishes the proof of the second part of Theorem 2.1.

B.2. Proof of $n = 3$, Theorem 3.1.

Proof. In this proof we suppress for typographical simplicity the shift superscripts (α, β) .

We write

$$g = \exp(\Gamma_c) \exp(\Gamma_d) \exp(\Gamma_e),$$

where

$$\Gamma_c = \text{Res}_z(C(z_1)E_{10}(z_1)), \quad \Gamma_d = \text{Res}_{z_1}(D(z_1)E_{20}(z_1)), \quad \Gamma_e = \text{Res}_{z_1}(E(z_1)E_{21}(z_1)).$$

This implies that $\tau_{k,l}$ is the sum of

$$(B.5) \quad c_{n,r,s} = \text{Res}_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \left(\prod_{i=1}^s C(x_i) \prod_{i=1}^r D(y_i) \prod_{i=1}^n E(z_i) p_{n,r,s} \right),$$

where

$$(B.6) \quad p_{n,r,s} = \langle T_1^k T_2^l v_0, \prod_{i=1}^s \psi_1^+(x_i) \psi_0^-(x_i) \prod_{i=1}^r \psi_2^+(y_i) \psi_0^-(y_i) \prod_{i=1}^n \psi_2^+(z_i) \psi_1^-(z_i) v_0 \rangle,$$

where

$$(B.7) \quad n + r = \ell, \quad s + r = k.$$

We can factorize this using Appendix D as

$$\begin{aligned} p_{n,r,s} = & (-1)^{\left(\frac{k(k-1)+l(l-1)}{2} + kl + \frac{s(s-1)+r(r-1)+n(n-1)}{2} + ns\right)} \times \\ & \times \langle Q_2^l v_0, \prod_{i=1}^r \psi_2^+(y_i) \prod_{i=1}^n \psi_2^+(z_i) v_0 \rangle \times \\ & \times \langle Q_1^{k-l} v_0, \prod_{i=1}^s \psi_1^+(x_i) \prod_{i=1}^n \psi_1^-(z_i) v_0 \rangle \langle Q_0^{-k} v_0, \prod_{i=1}^s \psi_0^-(x_i) \prod_{i=1}^r \psi_0^-(y_i) \rangle. \end{aligned}$$

Since, using (B.7),

$$(B.8) \quad (-1)^{\frac{k(k-1)+l(l-1)}{2} + kl + \frac{s(s-1)+r(r-1)+n(n-1)}{2} + ns} = (-1)^{\frac{r(r+1)}{2}},$$

the theorem follows from the calculation of correlation functions in Appendix E. \square

APPENDIX C. BIRKHOFF FACTORIZATION AND MATRIX ELEMENTS OF SEMI INFINITE WEDGE SPACE

In this Appendix we sketch proofs of Theorem 2.2 and 3.3. First we will discuss a more general statement about the Birkhoff factorization in $\widehat{\text{GL}}_n$.

C.1. Birkhoff Factorization and n -component Fermions. Let $\widetilde{\text{GL}}_{n-}$ be the subgroup of $\widetilde{\text{GL}}_n$ of elements $g_- = 1 + \mathcal{O}(z^{-1})$, and let $\widetilde{\text{GL}}_{n0+}$ be the subgroup of elements $g_{0+} = A + \mathcal{O}(z)$, where A is invertible (and independent of z). Then most elements in $\widetilde{\text{GL}}_n$ have a factorization

$$(C.1) \quad g = g_- g_{0+}, \quad g \in \widetilde{\text{GL}}_n.$$

We will express g_- in terms fermion matrix elements in semi-infinite wedge space $F^{(n)}$.

Recall that, just as the loop algebra $\widetilde{\mathfrak{gl}}_n$, the loop group $\widetilde{\text{GL}}_n$ does not actually act on $F^{(n)}$, we have a central extension (cf. [14])

$$1 \rightarrow \mathbb{C}^\times \rightarrow \widehat{\text{GL}}_n \xrightarrow{\pi} \widetilde{\text{GL}}_n \rightarrow 1,$$

and we have an action of $\widehat{\text{GL}}_n$ on $F^{(n)}$. There are subgroups $\widehat{\text{GL}}_{n-}$, $\widehat{\text{GL}}_{n0+}$ of $\widehat{\text{GL}}_n$ projecting under π to the corresponding subgroups of $\widetilde{\text{GL}}_n$; the image of \mathbb{C}^\times belongs to $\widehat{\text{GL}}_{n0+}$. Most elements $\hat{g} \in \widetilde{\text{GL}}_n$ will also have a Birkhoff factorization,

$$(C.2) \quad \hat{g} = \hat{g}_- \hat{g}_{0+}.$$

In fact the central extension is trivial over $\widehat{\text{GL}}_{n-}$ and we can and will identify $\widehat{\text{GL}}_{n-}$ with $\widetilde{\text{GL}}_{n-}$, and we will write g_- for either an element of $\widehat{\text{GL}}_{n-}$ or $\widetilde{\text{GL}}_{n-}$.

To calculate the negative component of g in the factorization (C.1) we choose a lift \hat{g} of g , i.e., $\pi(\hat{g}) = g$, and study the action of \hat{g} on $F^{(n)}$. If v_0 is the vacuum (A.4) of $F^{(n)}$, the τ -function is defined as the matrix element

$$(C.3) \quad \tau(\hat{g}) = \langle v_0, \hat{g} v_0 \rangle.$$

We have

$$\hat{g}_{0+} v_0 = \tau(\hat{g}) v_0,$$

and hence (assuming \hat{g} has a Birkhoff factorization, or $\tau(\hat{g}) \neq 0$)

$$(C.4) \quad g_- v_0 = g_- \hat{g}_{0+} v_0 / \tau(\hat{g}) = \hat{g} v_0 / \tau(\hat{g}).$$

Now write g_- in terms of matrix elements:

$$g_- = \sum_{a,b=0}^{n-1} g_{ab}(z) E_{ab},$$

where

$$(C.5) \quad g_{ab}(z) = \sum_{k \in \mathbb{Z}} g_{ab}^{(k)} z^{-k-1}, \quad g_{ab}^{(-1)} = \delta_{ab}, \quad g_{ab}^{(l)} = 0, \text{ if } l < -1.$$

We calculate, if $v_0 = e_0 \wedge e_1 \wedge e_2 \wedge \dots$,

$$g_- v_0 = (g_- e_0) \wedge (g_- e_1) \wedge (g_- e_2) \wedge \dots = v_0 + \sum_{k \geq 0} g_{ab}^{(k)} E_{ab} z^{-k-1} v_0 + \dots,$$

where the omitted terms are quadratic and higher in the E_{ab} . Now on $F^{(n)}$ (see section A.4)

$$(C.6) \quad E_{ab} z^{-k-1} =: \sum_l (e_a z^{l-k-1} \wedge) (i(e_b z^l)) =: \sum_{l \in \mathbb{Z}} {}_a \psi_{(l-k-1)b}^+ \psi_{(-l-1)}^-:$$

For $k \geq 0$ the normal ordering in (C.6) can be omitted and

$$g_{ab}^{(k)} E_{ab} z^{-k-1} v_0 = g_{ab}^{(k)} \left({}_a \psi_{(-k-1)b}^+ \psi_{(-1)}^- v_0 + {}_a \psi_{(1-k-1)b}^+ \psi_{(-2)}^- v_0 + \dots \right)$$

is a (finite) sum of elementary wedges ${}_a\psi_{(l-k-1)b}^+ \psi_{(-l-1)}^- v_0$, each with coefficient $g_{ab}^{(k)}$. To calculate $g_{ab}^{(k)}$ we just pick one of these elementary wedges, say the $l = 0$ term, and use orthogonality of elementary wedges to find ($k \geq 0$)

$$\begin{aligned} g_{ab}^{(k)} &= \langle {}_a\psi_{(-k-1)b}^+ \psi_{(-1)}^- v_0, g_- v_0 \rangle = \\ (C.7) \quad &= \langle {}_a\psi_{(-k-1)b}^+ \psi_{(-1)}^- v_0, \hat{g} v_0 \rangle / \tau(\hat{g}) = \\ &= \langle Q_b^{-1} v_0, {}_a\psi_{(k)}^- \hat{g} v_0 \rangle / \tau(\hat{g}), \end{aligned}$$

using (A.5) and (A.7).

Now observe that

$$: {}_a\psi_{(0)b}^+ \psi_{(-1)}^- : v_0 = \delta_{ab} v_0,$$

and

$$: {}_a\psi_{(l)b}^+ \psi_{(-1)}^- : v_0 = 0, \quad l > 0$$

This allows us to calculate $g_{ab}^{(k)}$ for $k < 0$ in the same way as for $k \geq 0$, see (C.7).

These remarks prove the following theorem.

Theorem C.1. *Let $g \in \widetilde{\text{GL}}_n$ admit a Birkhoff factorization $g = g_- g_{0+}$. Then*

$$g_- = \sum_{a,b=0}^{n-1} g_{ab}(z) E_{ab},$$

where

$$(C.8) \quad g_{ab}(z) = \langle Q_b^{-1} v_0, \psi_a^-(z) \hat{g} v_0 \rangle / \tau(\hat{g}),$$

and $\hat{g} \in \widehat{\text{GL}}_n$ is any lift of g , so that $\pi(\hat{g}) = g$. The τ function is given by (C.3).

C.2. The 2×2 -case; Proof of Theorem 2.2. Now we specialize in Theorem C.1 n and g : in this subsection $n = 2$ and

$$g = \pi(g^{[k](\alpha)}) = \pi(T^{-k}) \pi(g_C^{(\alpha)}),$$

where $\pi(g_C^{(\alpha)})$ is given by (2.3). In this case our calculation in Theorem C.1 of the Birkhoff factorization gives us

$$g_-^{[k](\alpha)} = \sum_{a,b=0}^1 g_{ab}^{[k](\alpha)}(z) E_{ab}, \quad g_{ab}^{[k](\alpha)}(z) = \langle Q_b^{-1} v_0, \psi_a^-(z) T^{-k} g_C^{(\alpha)} v_0 \rangle / \tau_k^{(\alpha)} E_{ab},$$

where the τ -function is defined in (2.6). We'll proceed to rewrite $g_{ab}^{[k](\alpha)}(z)$.

First of all, we will expand $g_C^{(\alpha)}$ in fermion operators. Note that

$$g_C^{(\alpha)} = \exp(\Gamma_C^{(\alpha)}),$$

where

$$(C.9) \quad \Gamma_C^{(\alpha)} = \text{Res}_{z_1} (C^{(\alpha)}(z_1) E_{10}(z_1)),$$

and $C^{(\alpha)}(z)$ is given by (2.2), and the generating series $E_{10}(z_1)$ by (A.16). This means that $g_C^{(\alpha)} = \sum_{l \geq 0} (\Gamma_C^{(\alpha)})^l / l!$, both acting on $H^{(2)}$ and on $F^{(2)}$. Hence

$$(C.10) \quad g_{ab}^{[k](\alpha)}(z) = \sum_{l \geq 0} \frac{1}{l!} \langle Q_b^{-1} v_0, \psi_a^-(z) T^{-k} (\Gamma_C^{(\alpha)})^l v_0 \rangle / \tau_k^{(\alpha)}$$

Next we use the standard grading on $F^{(2)}$. Note that $\Gamma_C^{(\alpha)}$ has degree $\delta_1 - \delta_0$ (since $E_{10}(z_1)$ has). So Q_b^{-1} has degree $-\delta_b$ and $\psi_a^-(z)T^{-k}(\Gamma_C^{(\alpha)})^l$ has degree $k(\delta_0 - \delta_1) + l(\delta_1 - \delta_0) - \delta_a$. Hence, by orthogonality of terms of different degree in $F^{(2)}$, we find that the only non zero contribution to the sum (C.10) arises when $l = k + a - b$ and

$$g_{ab}^{[k](\alpha)}(z) = \frac{1}{l!} \langle Q_b^{-1} v_0, \psi_a^-(z) T^{-k} (\Gamma_C^{(\alpha)})^l v_0 \rangle / \tau_k^{(\alpha)}.$$

From now on we will use often in formulas for $g_{ab}^{[k](\alpha)}(z)$ the abbreviation $l = k + a - b$.

Next we need to commute T^{-k} through $\psi_a^-(z)$.

Lemma C.2. *If $T = Q_1 Q_0^{-1}$ then*

$$\psi_a^-(z) T^{-k} = (-1)^k z^{k(2a-1)} T^{-k} \psi_a^-(z).$$

Proof. By (A.8), (A.9) we have

$$\begin{aligned} \psi_a^-(z) T^{-1} &= \psi_a^-(z) Q_0 Q_1^{-1} = \begin{cases} z^{-1} Q_0 \psi_a^-(z) Q_1^{-1} & \text{if } a = 0, \\ -Q_0 \psi_a^-(z) Q_1^{-1} & \text{if } a = 1. \end{cases} \\ &= \begin{cases} -z^{-1} T^{-1} \psi_a^-(z) & \text{if } a = 0, \\ -z T^{-1} \psi_a^-(z) & \text{if } a = 1. \end{cases} \\ &= -z^{2a-1} T^{-1} \psi_a^-(z). \end{aligned}$$

□

This implies that

$$\begin{aligned} g_{ab}^{[k](\alpha)}(z) &= \frac{(-1)^k z^{k(2a-1)}}{l!} \langle Q_b^{-1} v_0, T^{-k} \psi_a^-(z) (\Gamma_C^{(\alpha)})^l v_0 \rangle / \tau_k^{(\alpha)} = \\ (C.11) \quad &= \frac{(-1)^k z^{k(2a-1)}}{l!} \langle T^k Q_b^{-1} v_0, \psi_a^-(z) (\Gamma_C^{(\alpha)})^l v_0 \rangle / \tau_k^{(\alpha)}, \end{aligned}$$

by unitarity, (A.14).

Next we want to apply the factorization Lemma D.1. We need to write $T^k Q_b^{-1}$ in standard form $Q_1^\alpha Q_0^\beta$.

Lemma C.3.

$$T^k Q_b^{-1} = (-1)^{\frac{k(k-1)}{2}} (-1)^{bk} Q_1^{k-b} Q_0^{-k-1+b}$$

Proof. By Lemma A.2, Part 2

$$\begin{aligned} T^k Q_b^{-1} &= (-1)^{\frac{k(k-1)}{2}} Q_1^k Q_0^{-k} Q_b^{-1} = \\ &= (-1)^{\frac{k(k-1)}{2}} \begin{cases} Q_1^k Q_0^{-k-1} & \text{if } b = 0, \\ (-1)^k Q_1^{k-1} Q_0^{-k} & \text{if } b = 1. \end{cases} \end{aligned}$$

□

Now we are going to express $(\Gamma_C^{(\alpha)})^l$ in terms of fermion fields, see (C.9) and (A.17). Recall that $C^{(\alpha)} = \sum_{n \in \mathbb{Z}} c_{n+\alpha} z^{-n-1}$. Associated to the series $C^{(\alpha)}$ we have a map

$$c^{(\alpha)}: \mathbb{C}[[z, z^{-1}]] \rightarrow B = \mathbb{C}[[c_i]]_{i \in \mathbb{Z}}, \quad f(z) \mapsto \text{Res}_z(C^{(\alpha)}(z)f(z)).$$

We will need multiple copies of the map $c^{(\alpha)}$ acting on series in variables z_1, z_2, \dots . We put $c_i^{(\alpha)}(f(z_i)) = c^{(\alpha)}(f(z))$, for $i = 1, 2, \dots$, and impose linearity in the variables z_j , $j \neq i$, i.e., the condition that (for instance) $c_i^{(\alpha)}(f(z_i)z_j^m) = z_j^m c_i^{(\alpha)}(f(z_i))$.

Then we can write

$$\begin{aligned}
 (\Gamma_C^{(\alpha)})^l &= \prod_{i=1}^l c_i^{(\alpha)} \prod_{j=1}^l E_{10}(z_j) = \\
 &= \prod_{i=1}^l c_i^{(\alpha)} \prod_{j=1}^l \psi_1^+(z_j) \psi_0^-(z_j) = \\
 (C.12) \quad &= (-1)^{\frac{l(l-1)}{2}} \prod_{i=1}^l c_i^{(\alpha)} \prod_{s=1}^l \psi_1^+(z_s) \prod_{t=1}^l \psi_0^-(z_t).
 \end{aligned}$$

Here we use that in the expression of $E_{10}(z_1)$ in fermions we don't need normal ordering, and also Lemma A.2, part 2.

Now combine in (C.11) Lemma C.3 and (C.12) to get

$$g_{ab}^{[k](\alpha)}(z) = \epsilon_{ab}^{[k](\alpha)}(z) \prod_{i=1}^l c_i^{(\alpha)} \langle Q_1^{k-b} Q_0^{-k-1+b} v_0, \psi_a^-(z) \prod_{s=1}^l \psi_1^+(z_s) \prod_{t=1}^l \psi_0^-(z_t) v_0 \rangle / \tau_k^{(\alpha)},$$

where

$$\epsilon_{ab}^{[k](\alpha)}(z) = \frac{(-1)^k z^{k(2a-1)}}{l!} (-1)^{\frac{l(l-1)}{2}} (-1)^{\frac{k(k-1)}{2}} (-1)^{kb}.$$

Here still $l = k + a - b$. Now use the factorization Lemma D.1; we get

$$\begin{aligned}
 g_{ab}^{[k](\alpha)}(z) &= \frac{\epsilon_{ab}^{[k](\alpha)}(z)}{\tau_k^{(\alpha)}} \prod_{i=1}^l c_i \langle Q_1^{k-b} v_0, \left\{ \begin{array}{ll} \prod_{j=1}^l \psi_1^+(z_j) v_0 & a = 0, \\ \psi_1^-(z) \prod_{j=1}^l \psi_1^+(z_j) v_0 & a = 1 \end{array} \right\} \times \\
 &\times \langle Q_0^{-k-1+b} v_0, \left\{ \begin{array}{ll} \psi_0^-(z) \prod_{j=1}^l \psi_0^-(z_j) v_0 & a = 0, \\ \prod_{j=1}^l \psi_0^-(z_j) v_0 & a = 1 \end{array} \right\} \rangle
 \end{aligned}$$

Now using Lemma E.4 to calculate the factors involving $\psi_0(z)$ and $\psi_1(z)$ separately we find

$$g_{ab}^{[k](\alpha)}(z) = \frac{\epsilon_{ab}^{[k](\alpha)}(z)}{\tau_k^{(\alpha)}} \prod_{i=1}^l c_i \left(\det(V_{\{z_i\}}^{(l)})^2 \prod_{j=1}^l (z - z_j)^{1-2a} \right).$$

Comparing with the expression for the τ -function in Theorem 2.1 and the definition of the shift fields (2.10) gives Theorem 2.2.

C.3. Birkhoff Factorization in the 3×3 case, Proof of Theorem 3.3. The proof of Theorem 3.3 is similar to that of 2.2 sketch in the previous subsection. We leave the details to the reader.

APPENDIX D. FACTORIZATION AND REDUCTION TO ONE COMPONENT FERMIONS

Often we want to calculate a matrix element in $F^{(n)}$ of fermion fields of the form

$$\langle Q_{n-1}^{\alpha_{n-1}} \dots Q_1^{\alpha_1} Q_0^{\alpha_0} v_0, P(\psi_a^\pm(z_a)) v_0 \rangle,$$

where P is some polynomial in the fermion fields $\psi_a^\pm(z_a)$, $a = 0, 1, \dots, n-1$. By linearity we can reduce to the case where $P = M$ is a monomial, and then we can in the monomial

rearrange the factors as in Definition A.1: $M = M_{n-1} \dots M_1 M_0$, $M_a = M_a^+ M_a^-$, where M_a^\pm is a monomial in a single type of fermions, ordered according to the subscript of the arguments of the fields:

$$(D.1) \quad M_a^\pm = \overleftarrow{\prod}_{i=1}^t \psi_a^\pm(z_i) = \psi_a^\pm(z_t) \psi_a^\pm(z_{t-1}) \dots \psi_a^\pm(z_2) \psi_a^\pm(z_1).$$

This defines the ordered product of fermion fields.

We calculate such matrix elements using the following factorization Lemma.

Lemma D.1. *Let $M_a = M_a(\psi_a^\pm(z_a))$, $a = 0, \dots, n-1$, be monomials in fermion fields of type a . Then*

$$\langle Q_{n-1}^{\alpha_{n-1}} Q_1^{\alpha_1} Q_0^{\alpha_0} v_0, M_{n-1} \dots M_1 M_0 v_0 \rangle = \langle Q_{n-1}^{\alpha_{n-1}}, M_{n-1} v_0 \rangle \dots \langle Q_1^{\alpha_1}, M_1 v_0 \rangle \langle Q_0^{\alpha_0}, M_0 v_0 \rangle.$$

So the correlation functions we want to calculate reduce to products of correlation functions containing only one type of fermionic translation operator Q_a and one type of fermion fields $\psi_a^\pm(z)$. Such correlation functions (on $F^{(n)}$) are the same as the corresponding correlation function calculated on F , one component semi-infinite wedge space, according to the following Lemma.

Lemma D.2.

$$\langle Q_a^\alpha v_0, F(\psi_a^\pm) v_0 \rangle_{F^{(n)}} = \langle Q^\alpha v_0, F(\psi^\pm) v_0 \rangle_F.$$

In Appendix E we review some formulas for one component fermions.

APPENDIX E. ONE COMPONENT FERMION CORRELATION FUNCTIONS

In this Appendix we collect some results of one component fermions. In other words we are dealing with the fermionic Fock space $F = F^{(1)}$, based on $H = H^{(1)}$. The whole discussion of Appendix A transfers to the present one component context. For typographical convenience we will write $\psi^\pm(z)$ for $\psi_0^\pm(z)$ and similarly we write $Q^{\pm 1}$ for $Q_0^{\pm 1}$.

The correlation functions are matrix elements in $F = F^{(1)}$ of the form $\langle Q^k v_0, M(z_i, w_j) v_0 \rangle$, where M is some monomial in $\psi(z_i)$ and $\psi(w_j)$. For reasons of orthogonality of distinct charges we need to insert a power of the fermionic translation operator Q .

The simplest case of a correlation function is

$$\langle Q^{\pm 1} v_0, \psi^\pm(z) v_0 \rangle = \langle \psi_{(-1)}^\pm v_0, \sum_{k \in \mathbb{Z}} \psi_{(-k-1)}^\pm v_0 z^k \rangle = 1,$$

since only the $k = 0$ terms contributes.

For more general correlation functions we use the following Lemma recursively.

Lemma E.1. *Let $M(z_i, w_j)$ be a monomial in $\psi^+(z_i)$, $i = 1, 2, \dots, s$ and $\psi^-(w_j)$, $j = 1, 2, \dots, t$, and put $k = s - t$. Then*

$$\langle Q^{k \pm 1} v_0, \psi^\pm(z) M(z_i, w_j) v_0 \rangle = \frac{\prod_{i=1}^s (z - z_i)}{\prod_{j=1}^t (z - \underline{w_j})} \langle Q^{k \pm 1} v_0, M(z_i, w_j) v_0 \rangle,$$

where we expand in positive powers of the underlined variables.

For example, if all fermion fields are of the same type we get simple determinantal expressions for the correlation functions. Recall the ordered product of fermions, (D.1).

Lemma E.2. *For all $k \geq 1$ we have*

$$\langle Q^{\pm k} v_0, \overleftarrow{\prod}_{i=1}^k \psi^{\pm}(z_i) v_0 \rangle = \det(V_{\{z_i\}}^{(k)}).$$

Here the Vandermonde matrix in $\{z_i\} = \{z_1, z_2, \dots, z_k\}$ is given by

$$V_{\{z_i\}}^{(k)} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_k \\ z_1^2 & z_2^2 & \dots & z_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{k-1} & z_2^{k-1} & \dots & z_k^{k-1} \end{pmatrix} = (a_{ij})_{i,j=1}^k,$$

where $a_{ij} = z_j^{i-1}$. Then we have

$$\begin{aligned} \det(V_{\{z_i\}}^k) &= \sum_{\sigma \in S_k} (-1)^{|\sigma|} \prod_{i=1}^k a_{i\sigma(i)} = \\ &= \sum_{\sigma \in S_k} (-1)^{|\sigma|} \prod_{i=1}^k z_{\sigma(i)}^{i-1} = \\ &= \prod_{k \geq \alpha > \beta \geq 1} (z_{\alpha} - z_{\beta}). \end{aligned}$$

We need

Lemma E.3.

$$\begin{aligned} &\langle Q^{m-n} v_0, \psi^+(w_1) \dots \psi^+(w_m) \psi^-(y_1) \dots \psi^-(y_n) v_0 \rangle = \\ &= \frac{\prod_{1 \leq i < j \leq m} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (w_i - y_j)} \end{aligned}$$

Proof. We first consider the case that $m < n$. So

$$\begin{aligned} &\langle Q^{m-n} v_0, \psi^+(w_1) \dots \psi^+(w_m) \psi^-(y_1) \dots \psi^-(y_n) v_0 \rangle = \\ &= \langle \psi_{(m-n)}^- \psi_{(m-n+1)}^- \dots \psi_{(-1)}^- v_0, \psi^+(w_1) \dots \psi^+(w_m) \psi^-(y_1) \dots \psi^-(y_n) v_0 \rangle, \end{aligned}$$

which is equal to the sum of the coefficients corresponding to all ways of pulling out $\psi_{(m-n)}^- \psi_{(m-n+1)}^- \dots \psi_{(-1)}^- v_0$, from the product of fermionic fields acting on the vacuum,

$$\psi^+(w_1) \dots \psi^+(w_m) \psi^-(y_1) \dots \psi^-(y_n) v_0.$$

Given a $\sigma \in \mathfrak{S}_n$, we reorder the fermionic fields and record the sign obtained from doing so:

$$\text{sgn}(\sigma) \psi^+(w_1) \dots \psi^+(w_m) \psi^-(y_{\sigma(1)}) \dots \psi^-(y_{\sigma(n)}) v_0$$

We then pull out terms in such a way that no additional sign changes occur from permuting the operators: We pull out $\psi_{(-1)}^-$ from $\psi^-(y_{\sigma(n)})$, $\psi_{(-2)}^-$ from $\psi^-(y_{\sigma(n-1)})$, \dots , and $\psi_{(m-n)}^-$ from $\psi^-(y_{\sigma(m-n)})$. The product of the coefficients corresponding to these choices is $y_{\sigma(n)}^0 y_{\sigma(n-1)}^1 \dots y_{\sigma(m-n+1)}^{n-m-1}$. The remaining contributions for this choice of σ come from pulling out coefficients of products of wedging and contracting operators from

$$\psi^+(w_1) \dots \psi^+(w_m) \psi^-(y_{\sigma(1)}) \dots \psi^-(y_{\sigma(m)})$$

whose actions cancel with each other. We again pull out these terms such a way that no additional sign changes occur: We count only contributions coming from terms in $\psi^+(w_m)\psi^-(y_{\sigma(1)})$ that cancel with each other, terms in $\psi^+(w_{m-1})\psi^-(y_{\sigma(2)})$ that cancel with each other, \dots and terms in $\psi^+(w_1)\psi^-(y_{\sigma(m)})$ that cancel with each other.

We claim that we can count each pair, $\psi^+(w_{m-i})\psi^-(y_{\sigma(i+1)})$, $0 \leq i \leq m-1$, as contributing $\frac{1}{w_{m-i} - y_{\sigma(i+1)}} = \sum_{\ell=0}^{\infty} \frac{y_{\sigma(i+1)}^{\ell}}{w_{m-i}^{\ell+1}}$. We know we are not omitting any nontrivial terms in doing this, since any $\frac{y_{\sigma(i+1)}^{\ell}}{w_{m-i}^{\ell+1}}$ with $\ell < 0$ corresponds to $\psi_{(\ell)}^+\psi_{(-\ell-1)}^-$ and $\psi_{(-\ell-1)}^-$ kills the vacuum or any vector obtained by acting by contracting operators on the vacuum. We must therefore only prove that we are not including any extra nontrivial terms. Towards this end, consider some monomial,

$$\text{sgn}(\sigma)y_{\sigma(n)}^0 y_{\sigma(n-1)} \cdots y_{\sigma(m+1)}^{n-m-1} \frac{y_{\sigma(1)}^{\ell_1}}{w_m^{\ell_1+1}} \frac{y_{\sigma(2)}^{\ell_2}}{w_{m-1}^{\ell_2+1}} \cdots \frac{y_{\sigma(m)}^{\ell_m}}{w_1^{\ell_m+1}},$$

corresponding to a product of wedges acting on the vacuum vector which give 0. Since all of the wedges, $\psi_{(\ell)}^-$, is such that $\ell < 0$, the only way this is possible is if two of the wedges are the same. But this means that two of the y_i s in the above expression are being raised to the same power. Define a new element, $\gamma \in \mathfrak{S}_n$ by composing σ with the transposition that interchanges these two y_i s. The sign of this new element is $-\text{sgn}(\sigma)$. So there a monomial in the expansion of

$$\text{sgn}(\gamma) \frac{y_{\gamma(n)}^0 y_{\gamma(n-1)} \cdots y_{\gamma(m+1)}^{n-m-1}}{\prod_{i=0}^{m-1} (w_{m-i} - y_{\gamma(i+1)})}$$

which cancels with the above monomial.

Summing over all $\sigma \in \mathfrak{S}_n$, we have that

$$\begin{aligned} \langle Q^{m-n} v_0, \psi^+(w_1) \cdots \psi^+(w_m) \psi^-(y_1) \cdots \psi^-(y_n) v_0 \rangle &= \\ &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \frac{y_{\sigma(n)}^0 y_{\sigma(n-1)} \cdots y_{\sigma(m+1)}^{n-m-1}}{\prod_{i=0}^{m-1} (w_{m-i} - y_{\sigma(i+1)})}. \end{aligned}$$

Using Leibniz's formula to expand this as a determinant and then computing the determinant, we find that this is exactly equal to

$$\frac{\prod_{1 \leq i < j \leq m} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (w_i - y_j)}$$

The proof in the case that $m \geq n$ is similar. Here, we need to argue that

$$\begin{aligned} \langle Q^{m-n} v_0, \psi^+(w_1) \cdots \psi^+(w_m) \psi^-(y_1) \cdots \psi^-(y_n) v_0 \rangle &= \\ &= \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \frac{w_{\sigma(1)}^{m-n-1} w_{\sigma(2)}^{m-n-2} \cdots w_{\sigma(m-n)}^0}{\prod_{i=0}^{n-1} (w_{\sigma(m-i)} - y_{i+1})}. \end{aligned}$$

□

We also need the following case.

Lemma E.4.

$$\langle Q^{m-n-1}v_0, \psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i)v_0 \rangle =$$

$$\frac{\prod_{i=1}^m (z - y_i) \prod_{1 \leq i < j \leq m} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^n (z - w_i) \prod_{i=1}^m \prod_{j=1}^n (w_i - y_j)}$$

Proof. Consider the case that $m > n$. The proof here is analogous to LEMMA E.3.

$$\langle Q^{m-n-1}v_0, \psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i)v_0 \rangle$$

is the sum of the products of coefficients corresponding to all of the ways of pulling out $Q^{m-n-1}v_0 = \psi_{(m-n-1)}^+ \cdots \psi_{(-1)}^+ v_0$ from

$$\psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i)v_0.$$

Since $Q^{m-n-1}v_0$ is a product of wedging operators acting on the vacuum, the actions of the contracting operators in the negative fermion fields, $\psi^-(z)$ and the $\psi^-(y_i)$ s, must cancel with the actions of the wedging operators in the positive fermion fields, the $\psi^+(w_i)$ s. The remaining positive fermion fields contribute to the

$$\psi_{(m-n-1)}^+ \cdots \psi_{(-1)}^+ v_0.$$

Given an element $\sigma \in \mathfrak{S}_m$, we reorder the positive fermion fields and record the sign obtained by doing so:

$$\text{sgn}(\sigma) \psi^-(z) \psi^+(w_{\sigma(1)}) \cdots \psi^+(w_{\sigma(m)}) \psi^-(y_1) \cdots \psi^-(y_n) v_0.$$

We then pull out wedging and contracting operators in a fixed way such that no additional sign changes occur. Since $m > n$, taking the sum over all $\sigma \in \mathfrak{S}_m$, we obtain all terms that contribute to

$$\langle Q^{m-n-1}v_0, \psi^-(z) \psi^+(w_1) \cdots \psi^+(w_m) \psi^-(y_1) \cdots \psi^-(y_n) v_0 \rangle.$$

In more detail, given $\sigma \in \mathfrak{S}_m$, we have

$$\text{sgn}(\sigma) \psi^-(z) \psi^+(w_{\sigma(1)}) \cdots \psi^+(w_{\sigma(m)}) \psi^-(y_1) \cdots \psi^-(y_n) v_0,$$

and we collect all of the coefficients in $\psi^-(z) \psi^+(w_{\sigma(1)})$ corresponding to wedging and contracting operators whose actions cancel with each other. We pull out $\psi_{(n-m+1)}^+$ from $\psi^+(w_{\sigma(2)})$, $\psi_{(n-m)}^+$ from $\psi^+(w_{\sigma(3)})$, \cdots , and $\psi_{(-1)}^+$ from $\psi^+(w_{\sigma(m-n)})$. We then collect the coefficients corresponding to all pairs of wedging and contracting operators in $\psi^+(w_{\sigma(m)}) \psi^-(y_1)$ whose actions cancel with each other, all pairs of wedging and contracting operators in $\psi^+(w_{\sigma(m-1)}) \psi^-(y_2)$ whose actions cancel with each other, \cdots , and all pairs of wedging and contracting operators in $\psi^+(w_{\sigma(m-n+1)}) \psi^-(y_n)$ whose actions cancel with each other.

We claim that each pair, $\psi^+(a)\psi^-(b)$ corresponding to wedging and contracting operators whose actions cancel with each other can be thought of as contributing $\frac{1}{a-b}$. To see why this is true, we comment that each monomial in the expansion of

$$\langle Q^{m-m-1}v_0, \psi^-(z) \prod_{i=1}^m \psi^-(w_i) \prod_{i=1}^n \psi^-(y_i)v_0 \rangle$$

corresponds to a choice of fermionic operators. Given $\sigma \in \mathfrak{S}_m$, a monomial corresponds to a choice of k_i s such that

$$\text{sgn } \sigma \times \psi_{(-k_0-1)}^- \psi_{(k_0)}^+ \psi_{(n-m+1)}^+ \cdots \psi_{(-1)}^+ \psi_{(-k_1-1)}^+ \psi_{(-k_2-1)}^+ \cdots \psi_{(-k_n-1)}^+ \psi_{(k_n)}^- \psi_{(k_{n-1})}^- \cdots \psi_{(k_1)}^- v_0 = v_0.$$

We first note that k_1, \dots, k_n must all be less than 0, since if $k_i > 0$, the corresponding contracting operator, $\psi_{(k_i)}^-$ kills the vacuum or any vector obtained from the vacuum by acting by contracting operators. The only other possible way of choosing k_i s, $1 \leq i \leq n$, so that the above product of fermion operators acting on the vacuum is 0, is if two k_i s are equal or some k_i is $n-m \leq k_i \leq -1$. Using the same argument that we used in LEMMA E.3, we see that the monomials corresponding to such choices cancel with each other. So we are not including any extra terms in allowing each of the k_i s to range over all negative integers. So each pair, $\psi^+(w_{\sigma(m-i)})\psi^-(y_{i+1})$, $0 \leq i \leq n-1$, from which we pull out coefficients corresponding to wedging and contracting operators whose actions cancel with each other can be thought of as contributing $\frac{1}{w_{\sigma(m-i)} - y_{i+1}}$. Similarly, $k_0 < 0$ and even though choosing $k_0 = i$ for some $n-m+1 \leq i \leq -1$ will result in killing the vacuum, we can include such choices since the corresponding monomials all cancel with each other. Summing over all $\sigma \in \mathfrak{S}_m$, we obtain

$$\sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \frac{w_{\sigma(2)}^{m-n-2} \cdots w_{\sigma(m-n)}^0}{(z - w_{\sigma(1)})(w_{\sigma(m)} - y_1)(w_{\sigma(m-1)} - y_2) \cdots (w_{\sigma(m-n+1)} - y_n)},$$

which is precisely the Leibniz form of

$$\det \begin{bmatrix} \frac{1}{z-w_1} & \frac{1}{z-w_2} & \cdots & \frac{1}{z-w_m} \\ w_1^{m-n-2} & w_2^{m-n-2} & \cdots & w_m^{m-n-2} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \\ \frac{1}{w_1-y_n} & \frac{1}{w_2-y_n} & \cdots & \frac{1}{w_m-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_m-y_1} \end{bmatrix},$$

and the result holds by LEMMA G.3.

Next, consider the case that $m \leq n$. $Q^{m-n-1}v_0$ here is a product of contracting operators acting on the vacuum.

Since $m \leq n$, there are more negative fermion fields than positive fermion fields. We would like to proceed exactly as we did in the $m > n$ case. Ideally, we would want to fix an order of the $n+1$ negative fermion fields given by some permutation $\sigma \in \mathfrak{S}_{n+1}$, pull out (in a fixed way) wedging and contracting operators whose product acting on the vacuum is $Q^{m-n-1}v_0$, and sum over all permutations of the negative fermion fields to obtain all ways of pulling

out $Q^{m-n-1}v_0$ from $\psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i)v_0$. This cannot be done as easily as in the $m > n$ case though, since the positive and negative fermion fields do not commute and the positive fermion fields are sandwiched between $\psi^-(z)$ and $\prod_{i=1}^n \psi^-(y_i)$. We would thus obtain more than a sign change if we chose a permutation that did not fix $\psi^-(z)$.

By writing

$$\begin{aligned} \langle Q^{m-n-1}v_0, \psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i)v_0 \rangle = \\ = \langle v_0, \psi_{(0)}^+ \psi_{(1)}^+ \cdots \psi_{(n-m)}^+ \psi^-(z) \psi^+(w_1) \cdots \psi^+(w_m) \prod_{i=1}^n \psi^-(y_i)v_0 \rangle, \end{aligned}$$

we see that, in order to obtain a product of the vacuum vector from

$$\psi_{(0)}^+ \psi_{(1)}^+ \cdots \psi_{(n-m)}^+ \psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i)v_0,$$

the actions of any contracting operators in $\psi^-(z)$ must either cancel with the actions of wedging operators in the $\psi^+(w_i)$ s to the right of $\psi^-(z)$, or must cancel with the wedging operators, $\psi_{(i)}^+$, $0 \leq i \leq n-m$ to the left of $\psi^-(z)$. There are thus $n+1$ choices, $\psi_{(0)}^+$, $\psi_{(1)}^+$, \cdots , $\psi_{(n-m)}^+$, $\psi^+(y_1)$, \cdots , $\psi^+(y_n)$, for terms to which $\psi^-(z)$ can be paired. Given one of these $n+1$ choices, we then can proceed as in the $m > n$ case, fixing an order of negative fermion fields given by some $\sigma \in \mathfrak{S}_n$, pulling out fermion operators from these ordered terms in a fixed way, and reordering the corresponding monomials. Summing over all $\sigma \in \mathfrak{S}_n$ and all $n+1$ choices, we obtain our result.

Since the fermion operators in $\psi^-(z)$ can be paired only with the $\psi_{(i)}^+$ s to the left of $\psi^-(z)$ or with the $\psi_{(i)}^+$ s in the $\psi^+(w_i)$ s, our choices of fermion operators have only two possible forms:

(1)

$$\begin{aligned} & (-1)^{n-m-k_0} \operatorname{sgn} \sigma \times \\ & \langle v_0, \psi_{(0)}^+ \psi_{(1)}^+ \cdots \psi_{(k_0-1)}^+ \psi_{(k_0+1)}^+ \cdots \psi_{(n-m)}^+ \psi_{(k_0)}^+ \\ & \psi_{(-k_0-1)}^- \psi_{(k_1)}^+ \psi_{(k_2)}^+ \cdots \psi_{(k_m)}^+ \psi_{(-k_m-1)}^- \cdots \psi_{(-k_1-1)}^- \psi_{(m-n-1)}^- \cdots \psi_{(-k_0-2)}^- \psi_{(-k_0)}^- \cdots \psi_{(-2)}^- \psi_{(-1)}^- v_0 \rangle \end{aligned}$$

(The additional sign appearing in front of $\operatorname{sgn} \sigma$ comes from shifting the $\psi_{(k_0)}^+$, $0 \leq k_0 \leq n-m$ past the $n-m-k_0$ $\psi_{(i)}^+$ s to its right, so that it appears immediately next to $\psi^-(z)$.)

(2)

$$\begin{aligned} & (-1)^{i-1} \operatorname{sgn} \sigma \times \\ & \langle v_0, \psi_{(0)}^+ \psi_{(1)}^+ \cdots \psi_{(n-m)}^+ \psi_{(-k_i-1)}^- \psi_{(k_i)}^+ \psi_{(k_1)}^+ \cdots \psi_{(k_{i-1})}^+ \psi_{(k_{i+1})}^+ \cdots \psi_{(k_m)}^+ \\ & \psi_{(-k_m-1)}^- \psi_{(-k_{m-1}-1)}^- \cdots \psi_{(-k_{i+1}-1)}^- \psi_{(-k_{i-1}-1)}^- \cdots \psi_{(-k_1-1)}^- \psi_{(m-n-1)}^- \cdots \psi_{(-2)}^- \psi_{(-1)}^- v_0 \rangle \end{aligned}$$

(Here, the additional sign appearing in front of $\operatorname{sgn} \sigma$ comes from shifting the $\psi^+(w_i)$ past the $i-1$ $\psi_{(w_j)}^+$ s to its left so that $\psi^+(w_i)$ appears next to $\psi^-(z)$.)

We now argue which choices of k_i s contribute to $\langle Q^{m-n-1}v_0, \psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i)v_0 \rangle$.

For choices of the form (1), each k_j $1 \leq j \leq m$ must be nonnegative, since if $k < 0$, $\psi_{(-k-1)}^-$ kills the vacuum or any vector obtained from the vacuum by acting by contracting operators. Given k_0 , $0 \leq k_0 \leq n - m$,

$$\psi_{(-k_m-1)}^- \cdots \psi_{(-k_1-1)}^- \psi_{(m-n-1)}^- \cdots \psi_{(-k_0-2)}^- \psi_{(-k_0)}^- \cdots \psi_{(-1)}^- v_0 = 0$$

if any of the k_j s, $1 \leq j \leq m$, $j \neq k_0$ are equal to each other or to some ℓ , $0 \leq \ell \leq n - m$. Using the same argument used above and in LEMMA E.3 though, any such monomials cancel with each other. So we can sum over all $k_j \geq 0$, $j \neq i$. Then summing over all k_0 , $0 \leq k_0 \leq n - m$ and all $\sigma \in \mathfrak{S}_n$, we obtain

$$\sum_{k_0=0}^{n-m} (-1)^{n-m-k_0} z^{k_0} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn } \sigma \frac{y_{\sigma(m+1)}^{n-m} \cdots y_{\sigma(n-k_0)}^{k_0+1} y_{\sigma(n+1-k_0)}^{k_0-1} \cdots y_{\sigma(n)}^0}{(w_m - y_{\sigma(1)})(w_{m-1} - y_{\sigma(2)}) \cdots (w_1 - y_{\sigma(m)})}.$$

We next sum over all possible values of k_i s for choices of the form (2), making sure not to include any choices equivalent to those already counted in (1).

When $k < 0$, $\psi_{(-k-1)}^-$ kills the vacuum or any vector obtained from the vacuum by acting by contracting operators. Given a choice of i , we have

$$\begin{aligned} & \psi_{(k_1)}^+ \cdots \psi_{(k_{i-1})}^+ \psi_{(k_{i+1})}^+ \cdots \psi_{(k_m)}^+ \\ & \psi_{(-k_m-1)}^- \psi_{(-k_{m-1}-1)}^- \cdots \psi_{(-k_{i+1}-1)}^- \psi_{(-k_{i-1}-1)}^- \cdots \psi_{(-k_1-1)}^- \psi_{(m-n-1)}^- \cdots \psi_{(-2)}^- \psi_{(-1)}^- v_0 = \\ & = \psi_{(m-n-1)}^- \cdots \psi_{(-2)}^- \psi_{(-1)}^- v_0. \end{aligned}$$

So we can choose $k_i < 0$ or $0 \leq k_i \leq n - m$. We claim though, that $0 \leq k_i \leq n - m$ correspond to choices of fermion operators already made in (1). Note that the $k_i < 0$ cannot correspond to choices from (1) since k_i is the power of z in a given monomial and every monomial coming from (1) has power of z greater than or equal to 0 and less than or equal to $n - m$. As in our previous arguments, given a choice of k_i , we can allow our k_j s, $j \neq i$ to range over all nonnegative integers, since any terms corresponding to choices of k_j s that kill the vacuum will cancel with each other. Summing over all choices of k_i s we obtain

$$\sum_{i=1}^m (-1)^{i-1} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn } \sigma \frac{y_{\sigma(m)}^{n-m} y_{\sigma(m+1)}^{n-m-1} \cdots y_{\sigma(n)}^0}{(z - w_i)(w_1 - y_{\sigma(m-1)}) \cdots (w_{i-1} - y_{\sigma(m-i+1)})(w_{i+1} - y_{\sigma(m-i)}) \cdots (w_m - y_{\sigma(1)})}.$$

(To see why choices of k_i , $0 \leq k_i \leq n - m$ correspond to choices already made in (1), note that, up to a sign, the choices corresponding to (1) give monomials

$$\frac{z^{k_0} y_{\sigma(1)}^{k_m} \cdots y_{\sigma(m)}^{k_1} y_{\sigma(m+1)}^{n-m} \cdots y_{\sigma(m-k_0)}^{k_0+1} y_{\sigma(m-k_0+1)}^{k_0-1} \cdots y_{\sigma(n-1)} y_{\sigma(n)}^0}{w_1^{k_1+1} w_2^{k_2+1} \cdots w_m^{k_m+1}}$$

where $0 \leq k_0 \leq n - m$ and the remaining k_i s are nonnegative integers.

The choices corresponding to (2) give

$$\frac{z^{k_i} y_{\sigma(1)}^{k_m} \cdots y_{\sigma(m-i)}^{k_{i+1}} y_{\sigma(m-i+1)}^{k_{i-1}} \cdots y_{\sigma(m-1)}^{k_1} y_{\sigma(m)}^{n-m} \cdots y_{\sigma(n-1)} y_{\sigma(n)}^0}{w_1^{k_1+1} w_2^{k_2+1} \cdots w_m^{k_m+1}}$$

where $k_i < 0$ or $0 \leq k_i \leq n - m$ and the remaining k_j s range over all nonnegative integers. We comment that we do not need to know the signs for these monomials, since we can read from them the corresponding choice of fermion operators and this choice determines the sign. Thus, if any monomials corresponding to (1) or (2) are the same up to a sign, their corresponding signs must also match. Given a monomial of the form (2), whose k_i is $0 \leq k_i \leq n - m$, we can set $k_i = k_0$ for some k_0 corresponding to a choice in (1). If the choice of k_1, \dots, k_m in (1) also agree with those in (2), the powers of each w_i s in these monomials match and the powers of the y_i s match up to a permutation of their indices. We can then modify (1) by choosing a $\sigma \in \mathfrak{S}_n$ such that the monomials match exactly. So choices of k_i , $0 \leq k_i \leq n - m$ in (2) correspond to terms already counted in (1).)

Summing the contributions from (1) and (2), we have

$$\begin{aligned} & \langle Q^{m-n-1} v_0, \psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i) v_0 \rangle = \\ & \sum_{k_0=0}^{n-m} (-1)^{n-m-k_0} z^{k_0} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \frac{y_{\sigma(m+1)}^{n-m} \cdots y_{\sigma(n-k_0)}^{k_0+1} y_{\sigma(n+1-k_0)}^{k_0-1} \cdots y_{\sigma(n)}^0}{(w_m - y_{\sigma(1)})(w_{m-1} - y_{\sigma(2)}) \cdots (w_1 - y_{\sigma(m)})} + \\ & \sum_{i=1}^m (-1)^{i-1} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \frac{y_{\sigma(m)}^{n-m} y_{\sigma(m+1)}^{n-m-1} \cdots y_{\sigma(n)}^0}{(z - w_i)(w_1 - y_{\sigma(m-1)}) \cdots (w_{i-1} - y_{\sigma(m-i+1)})(w_{i+1} - y_{\sigma(m-i)}) \cdots (w_m - y_{\sigma(1)})}. \end{aligned}$$

This then gives us that

$$\begin{aligned} & \langle Q^{m-n-1} v_0, \psi^-(z) \prod_{i=1}^m \psi^+(w_i) \prod_{i=1}^n \psi^-(y_i) v_0 \rangle = \\ & (-1)^m \det \begin{bmatrix} \frac{-1}{z - w_m} & \frac{1}{w_m - y_1} & \frac{1}{w_m - y_2} & \cdots & \frac{1}{w_m - y_n} \\ \frac{z - w_{m-1}}{-1} & \frac{w_{m-1} - y_1}{1} & \frac{w_{m-1} - y_2}{1} & \cdots & \frac{w_{m-1} - y_n}{1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{-1}{z - w_2} & \frac{w_2 - y_1}{1} & \frac{w_2 - y_2}{1} & \cdots & \frac{w_2 - y_n}{1} \\ \frac{z - w_1}{-1} & \frac{w_1 - y_1}{1} & \frac{w_1 - y_2}{1} & \cdots & \frac{w_1 - y_n}{1} \\ z^{n-m} & y_1^{n-m} & y_2^{n-m} & \cdots & y_n^{n-m} \\ z^{n-m-1} & y_1^{n-m-1} & y_2^{n-m-1} & \cdots & y_n^{n-m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ z & y_1 & y_2 & \cdots & y_n \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}. \end{aligned}$$

(To see this, expand the determinant along the first column and express the corresponding minors using the Leibniz formula for the determinant.) The result then holds by LEMMA G.4. \square

APPENDIX F. PROOF OF DETERMINANTAL IDENTITIES

The proofs here are very similar to the proof of the Cauchy determinant identity. In each of the proofs, we perform elementary row and column operations, pull terms out, and show that the determinant reduces to some product times a Cauchy determinant.

Here, we will use the convention that $\frac{1}{a-b} = \sum_{i=0}^{\infty} \frac{b^i}{a^{i+1}}$. For example, when we write $\frac{-1}{a-b}$, we mean $-\sum_{i=0}^{\infty} \frac{b^i}{a^{i+1}}$ and not $\frac{1}{b-a} = \sum_{i=0}^{\infty} \frac{a^i}{b^{i+1}}$.

Lemma F.1. *When $m \geq n$,*

$$\det \begin{bmatrix} \frac{1}{w_n-y_1} & \frac{1}{w_n-y_2} & \cdots & \frac{1}{w_n-y_m} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_1-y_2} & \cdots & \frac{1}{w_1-y_m} \\ y_1^{m-n-1} & y_2^{m-n-1} & \cdots & y_m^{m-n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \frac{\prod_{1 \leq i < j \leq m} (y_i - y_j) \prod_{1 \leq i < j \leq n} (w_i - w_j)}{\prod_{i=1}^n \prod_{j=1}^m (w_i - y_j)}.$$

Proof. We first comment that if $m = n$, we obtain

$$\det \begin{bmatrix} \frac{1}{w_n-y_1} & \frac{1}{w_n-y_2} & \cdots & \frac{1}{w_n-y_n} \\ \frac{1}{w_{n-1}-y_1} & \frac{1}{w_{n-1}-y_2} & \cdots & \frac{1}{w_{n-1}-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_1-y_2} & \cdots & \frac{1}{w_1-y_n} \end{bmatrix}$$

and the result holds by the formula for a Cauchy determinant. So in the following, we may assume that $m > n$.

We subtract the last column from each of the previous columns, expand the determinant of the resulting matrix along the last row, and then pull out $\frac{\prod_{i=1}^{m-1} (y_i - y_m)}{\prod_{i=1}^n (w_i - y_m)}$. This gives us

$$\frac{\prod_{i=1}^{m-1} (y_i - y_m)}{\prod_{i=1}^n (w_i - y_m)} \det \begin{bmatrix} \frac{1}{w_n - y_1} & \frac{1}{w_n - y_2} & \cdots & \frac{1}{w_n - y_{m-1}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_1 - y_2} & \cdots & \frac{1}{w_1 - y_{m-1}} \\ \sum_{j=0}^{m-n-2} y_1^{m-n-2-j} y_m^j & \sum_{j=0}^{m-n-2} y_2^{m-n-2-j} y_m^j & \cdots & \sum_{j=0}^{m-n-2} y_3^{m-n-2-j} y_m^j y_m^{m-n-1} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^2 + y_m^2 + y_1 y_m & y_2^2 + y_m^2 + y_2 y_m & \cdots & y_{m-1}^2 + y_m^2 + y_{m-1} y_m \\ y_1 + y_m & y_2 + y_m & \cdots & y_{m-1} + y_m \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We then subtract y_m times the $(n+2)$ nd row from the $(n+1)$ st row, subtract y_m times the $(n+3)$ rd row from the $(n+2)$ nd row, \cdots , and subtract y_m times the last row from the second to last row. Since $\sum_{j=0}^{\ell} y_i^{\ell-j} y_m^j - \sum_{j=0}^{\ell-1} y_i^{\ell-1-j} y_m^{j+1} = y_i^{\ell}$, this gives us

$$\frac{\prod_{i=1}^{m-1} (y_i - y_m)}{\prod_{i=1}^n (w_i - y_m)} \det \begin{bmatrix} \frac{1}{w_n - y_1} & \frac{1}{w_n - y_2} & \cdots & \frac{1}{w_n - y_{m-1}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_1 - y_2} & \cdots & \frac{1}{w_1 - y_{m-1}} \\ y_1^{m-n-2} & y_2^{m-n-2} & \cdots & y_{m-1}^{m-n-2} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^2 & y_2^2 & \cdots & y_{m-1}^2 \\ y_1 & y_2 & \cdots & y_{m-1} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We then repeat this process $m-n-1$ more times, obtaining

$$\frac{\prod_{j=n+1}^m \prod_{i < j} (y_i - y_j)}{\prod_{i=1}^n \prod_{j=n+1}^m (w_i - y_j)} \det \begin{bmatrix} \frac{1}{w_n - y_1} & \frac{1}{w_n - y_2} & \cdots & \frac{1}{w_n - y_n} \\ \frac{1}{w_{n-1} - y_1} & \frac{1}{w_{n-1} - y_2} & \cdots & \frac{1}{w_{n-1} - y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_1 - y_2} & \cdots & \frac{1}{w_1 - y_n} \end{bmatrix}.$$

Since the determinant here is a Cauchy determinant equal to

$$\frac{\prod_{1 \leq i < j \leq n} (y_i - y_j) \prod_{1 \leq i < j \leq n} (w_i - w_j)}{\prod_{i=1}^n \prod_{j=1}^n (w_i - y_j)},$$

we have that the determinant with which we started is equal to

$$\begin{aligned}
& \frac{\prod_{j=n+1}^m \prod_{i < j} (y_i - y_j) \prod_{1 \leq i < j \leq n} (y_i - y_j) \prod_{1 \leq i < j \leq n} (w_i - w_j)}{\prod_{i=1}^n \prod_{j=n+1}^m (w_i - y_j) \prod_{i=1}^n \prod_{j=1}^n (w_i - y_j)} = \\
& = \frac{\prod_{1 \leq i < j \leq n} (w_i - w_j) \prod_{1 \leq i < j \leq m} (y_i - y_j)}{\prod_{i=1}^n \prod_{j=1}^m (w_i - y_j)}.
\end{aligned}$$

□

Lemma F.2. When $m < n$,

$$\det \begin{bmatrix} w_1^{n-m-1} & w_2^{n-m-1} & \cdots & w_{n-1}^{n-m-1} & w_n^{n-m-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ w_1 & w_2 & \cdots & w_{n-1} & w_n \\ 1 & 1 & \cdots & 1 & 1 \\ \frac{1}{w_1 - y_m} & \frac{1}{w_2 - y_m} & \cdots & \frac{1}{w_{n-1} - y_m} & \frac{1}{w_n - y_m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_2 - y_1} & \cdots & \frac{1}{w_{n-1} - y_1} & \frac{1}{w_n - y_1} \end{bmatrix} = \frac{\prod_{1 \leq i < j \leq m} (y_i - y_j) \prod_{1 \leq i < j \leq n} (w_i - w_j)}{\prod_{i=1}^n \prod_{j=1}^m (w_i - y_j)}.$$

Proof. We prove this identity in the same way that we proved our previous identity, by applying elementary row and column operations to reduce the determinant to a product times a Cauchy determinant. We begin by subtracting the last column from each of the previous columns. This gives us

$$\det \begin{bmatrix} w_1^{n-m-1} - w_n^{n-m-1} & w_2^{n-m-1} - w_n^{n-m-1} & \cdots & w_{n-1}^{n-m-1} - w_n^{n-m-1} & w_n^{n-m-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ w_1 - w_n & w_2 - w_n & \cdots & w_{n-1} - w_n & w_n \\ 0 & 0 & \cdots & 0 & 1 \\ \frac{-(w_1 - w_n)}{(w_1 - y_m)(w_n - y_m)} & \frac{-(w_2 - w_n)}{(w_2 - y_m)(w_n - y_m)} & \cdots & \frac{-(w_{n-1} - w_n)}{(w_{n-1} - y_m)(w_n - y_m)} & \frac{1}{w_n - y_m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{-(w_1 - w_n)}{(w_1 - y_1)(w_n - y_1)} & \frac{-(w_2 - w_n)}{(w_2 - y_1)(w_n - y_1)} & \cdots & \frac{-(w_{n-1} - w_n)}{(w_{n-1} - y_1)(w_n - y_1)} & \frac{1}{w_n - y_1} \end{bmatrix}.$$

We then expand along the $(n-m)$ th row to obtain

$$\det \begin{bmatrix} w_1^{n-m-1} - w_n^{n-m-1} & w_2^{n-m-1} - w_n^{n-m-1} & \cdots & w_{n-1}^{n-m-1} - w_n^{n-m-1} \\ \vdots & \vdots & \cdots & \vdots \\ w_1 - w_n & w_2 - w_n & \cdots & w_{n-1} - w_n \\ \frac{w_1 - w_n}{(w_1 - y_m)(w_n - y_m)} & \frac{w_2 - w_n}{(w_2 - y_m)(w_n - y_m)} & \cdots & \frac{w_{n-1} - w_n}{(w_{n-1} - y_m)(w_n - y_m)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{w_1 - w_n}{(w_1 - y_1)(w_n - y_1)} & \frac{w_2 - w_n}{(w_2 - y_1)(w_n - y_1)} & \cdots & \frac{w_{n-1} - w_n}{(w_{n-1} - y_1)(w_n - y_1)} \end{bmatrix}.$$

(We comment that expanding along the $(n - m)$ th row gives us a sign of $(-1)^m$. This sign cancels with the factors of -1 pulled out from each of our last m rows.) We then pull out

$$\frac{\prod_{i=1}^{n-1} (w_i - w_n)}{\prod_{i=1}^m (w_n - y_i)} \text{ to obtain } \frac{\prod_{i=1}^{n-1} (w_i - w_n)}{\prod_{i=1}^m (w_n - y_i)} \det \begin{bmatrix} \sum_{j=0}^{n-m-2} w_1^{n-m-2-j} w_n^j & \sum_{j=0}^{n-m-2} w_2^{n-m-2-j} w_n^j & \cdots & \sum_{j=0}^{n-m-2} w_{n-1}^{n-m-2-j} w_n^j \\ \vdots & \vdots & \cdots & \vdots \\ w_1 + w_n & w_2 + w_n & \cdots & w_{n-1} + w_n \\ 1 & 1 & \cdots & 1 \\ \frac{1}{w_1 - y_m} & \frac{1}{w_2 - y_m} & \cdots & \frac{1}{w_{n-1} - y_m} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_2 - y_1} & \cdots & \frac{1}{w_{n-1} - y_1} \end{bmatrix}.$$

Similarly to the proof of our previous identity, we subtract w_n times row 2 from row 1, subtract w_n times row 3 from row 2, \cdots , and subtract w_n times row $n - m - 1$ from row $n - m - 2$. This gives

$$\frac{\prod_{i=1}^{n-1} (w_i - w_n)}{\prod_{i=1}^m (w_n - y_i)} \det \begin{bmatrix} w_1^{n-m-2} & w_2^{n-m-2} & \cdots & w_{n-1}^{n-m-2} \\ \vdots & \vdots & \cdots & \vdots \\ w_1 & w_2 & \cdots & w_{n-1} \\ 1 & 1 & \cdots & 1 \\ \frac{1}{w_1 - y_m} & \frac{1}{w_2 - y_m} & \cdots & \frac{1}{w_{n-1} - y_m} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_2 - y_1} & \cdots & \frac{1}{w_{n-1} - y_1} \end{bmatrix}.$$

We then repeat the above process $n - m - 1$ more times, obtaining

$$\frac{\prod_{j=m+1}^n \prod_{i < j} (w_i - w_j)}{\prod_{i=m+1}^n \prod_{j=1}^m (w_i - y_j)} \det \begin{bmatrix} \frac{1}{w_1 - y_m} & \frac{1}{w_2 - y_m} & \cdots & \frac{1}{w_m - y_m} \\ \frac{1}{w_1 - y_{m-1}} & \frac{1}{w_2 - y_{m-1}} & \cdots & \frac{1}{w_m - y_{m-1}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_2 - y_1} & \cdots & \frac{1}{w_m - y_1} \end{bmatrix}.$$

The Cauchy determinant here is equal to

$$\frac{\prod_{1 \leq i < j \leq m} (w_i - w_j) \prod_{1 \leq i < j \leq m} (y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^m (w_i - y_j)},$$

so our original determinant is equal to

$$\frac{\prod_{1 \leq i < j \leq n} (w_i - w_j) \prod_{1 \leq i < j \leq m} (y_i - y_j)}{\prod_{i=1}^n \prod_{j=1}^m (w_i - y_j)}.$$

□

Lemma F.3. *When $m > n$,*

$$\det \begin{bmatrix} \frac{1}{z-w_1} & \frac{1}{z-w_2} & \cdots & \frac{1}{z-w_m} \\ w_1^{m-n-2} & w_2^{m-n-2} & \cdots & w_m^{m-n-2} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \\ \frac{1}{w_1-y_n} & \frac{1}{w_2-y_n} & \cdots & \frac{1}{w_m-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_m-y_1} \end{bmatrix} = \frac{\prod_{i=1}^n (z - y_i) \prod_{1 \leq i < j \leq m} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^m (z - w_i) \prod_{i=1}^m \prod_{j=1}^n (w_i - y_j)}.$$

Proof. If $m = n + 1$, our matrix is of the form

$$\begin{bmatrix} \frac{1}{z-w_1} & \frac{1}{z-w_2} & \cdots & \frac{1}{z-w_{n+1}} \\ \frac{1}{w_1-y_n} & \frac{1}{w_2-y_n} & \cdots & \frac{1}{w_{n+1}-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_{n+1}-y_1} \end{bmatrix},$$

and we can skip to **Step 2** (see below). If not, $m > n + 1$ and we proceed with **Step 1**.

Step 1: We subtract the last column from each of the previous columns to obtain

$$\det \begin{bmatrix} \frac{w_1-w_m}{(z-w_1)(z-w_m)} & \frac{w_2-w_m}{(z-w_2)(z-w_m)} & \cdots & \frac{w_{m-1}-w_m}{(z-w_{m-1})(z-w_m)} & \frac{1}{z-w_m} \\ w_1^{m-n-2} - w_m^{m-n-2} & w_2^{m-n-2} - w_m^{m-n-2} & \cdots & w_{m-1}^{m-n-2} - w_m^{m-n-2} & w_m^{m-n-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ w_1 - w_m & w_2 - w_m & \cdots & w_{m-1} - w_m & w_m \\ 0 & 0 & \cdots & 0 & 1 \\ \frac{-(w_1-w_m)}{(w_1-y_n)(w_m-y_n)} & \frac{-(w_2-w_m)}{(w_2-y_n)(w_m-y_n)} & \cdots & \frac{-(w_{m-1}-w_m)}{(w_{m-1}-y_n)(w_m-y_n)} & \frac{1}{w_m-y_n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{-(w_1-w_m)}{(w_1-y_1)(w_m-y_1)} & \frac{-(w_2-w_m)}{(w_2-y_1)(w_m-y_1)} & \cdots & \frac{-(w_{m-1}-w_m)}{(w_{m-1}-y_1)(w_m-y_1)} & \frac{1}{w_m-y_1} \end{bmatrix}.$$

We then expand along the $(m - n)$ th row to obtain

$$\det \begin{bmatrix} \frac{w_1 - w_m}{(z - w_1)(z - w_m)} & \frac{w_2 - w_m}{(z - w_2)(z - w_m)} & \cdots & \frac{w_{m-1} - w_m}{(z - w_{m-1})(z - w_m)} \\ w_1^{m-n-2} - w_m^{m-n-2} & w_2^{m-n-2} - w_m^{m-n-2} & \cdots & w_{m-1}^{m-n-2} - w_m^{m-n-2} \\ \vdots & \vdots & \cdots & \vdots \\ w_1 - w_m & w_2 - w_m & \cdots & w_{m-1} - w_m \\ \frac{w_1 - w_m}{(w_1 - y_n)(w_m - y_n)} & \frac{w_2 - w_m}{(w_2 - y_n)(w_m - y_n)} & \cdots & \frac{w_{m-1} - w_m}{(w_{m-1} - y_n)(w_m - y_n)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{w_1 - w_m}{(w_1 - y_1)(w_m - y_1)} & \frac{w_2 - w_m}{(w_2 - y_1)(w_m - y_1)} & \cdots & \frac{w_{m-1} - w_m}{(w_{m-1} - y_1)(w_m - y_1)} \end{bmatrix}.$$

(We obtain a $(-1)^n$ from expanding the determinant along the $(m - n)$ th row, but this sign cancels with the factors of -1 coming from the last n rows.)

$$\text{We then pull out } \frac{\prod_{i=1}^{m-1} (w_i - w_m)}{(z - w_m) \prod_{i=1}^n (w_m - y_i)} \text{ and write}$$

$$\frac{\prod_{i=1}^{m-1} (w_i - w_m)}{(z - w_m) \prod_{i=1}^n (w_m - y_i)} \det \begin{bmatrix} \frac{1}{z - w_1} & \frac{1}{z - w_2} & \cdots & \frac{1}{z - w_{m-1}} \\ \sum_{j=0}^{m-n-3} w_1^{m-n-3-j} w_m^j & \sum_{j=0}^{m-n-3} w_2^{m-n-3-j} w_m^j & \cdots & \sum_{j=0}^{m-n-3} w_{m-1}^{m-n-3-j} w_m^j \\ \vdots & \vdots & \cdots & \vdots \\ w_1 + w_m & w_2 + w_m & \cdots & w_{m-1} + w_m \\ 1 & 1 & \cdots & 1 \\ \frac{1}{w_1 - y_n} & \frac{1}{w_2 - y_n} & \cdots & \frac{1}{w_{m-1} - y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{1}{w_2 - y_1} & \cdots & \frac{1}{w_{m-1} - y_1} \end{bmatrix}.$$

We subtract w_m times the 3rd row from the 2nd row, subtract w_m times the 4th row from the 3rd row, \cdots , and subtract w_m times the $(m - n - 1)$ th row from the $(m - n - 2)$ nd row. This gives

$$\frac{\prod_{i=1}^{m-1} (w_i - w_m)}{(z - w_m) \prod_{i=1}^n (w_m - y_i)} \det \begin{bmatrix} \frac{1}{z-w_1} & \frac{1}{z-w_2} & \cdots & \frac{1}{z-w_{m-1}} \\ w_1^{m-n-3} & w_2^{m-n-3} & \cdots & w_{m-1}^{m-n-3} \\ \vdots & \vdots & \cdots & \vdots \\ w_1 & w_2 & \cdots & w_{m-1} \\ 1 & 1 & \cdots & 1 \\ \frac{1}{w_1-y_n} & \frac{1}{w_2-y_n} & \cdots & \frac{1}{w_{m-1}-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_{m-1}-y_1} \end{bmatrix}.$$

We repeat the above process $m - n - 2$ more times, obtaining:

Step 2:

$$\frac{\prod_{j=n+2}^m \prod_{i < j} (w_i - w_j)}{\prod_{i=n+2}^m (z - w_i) \prod_{i=n+2}^m \prod_{j=1}^n (w_i - y_j)} \det \begin{bmatrix} \frac{1}{z-w_1} & \frac{1}{z-w_2} & \cdots & \frac{1}{z-w_{n+1}} \\ \frac{1}{w_1-y_n} & \frac{1}{w_2-y_n} & \cdots & \frac{1}{w_{n+1}-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_{n+1}-y_1} \end{bmatrix}.$$

To simplify this determinant, add the 1st row to all other rows.

$$\begin{aligned} & \det \begin{bmatrix} \frac{1}{z-w_1} & \frac{1}{z-w_2} & \cdots & \frac{1}{z-w_{n+1}} \\ \frac{1}{w_1-y_n} & \frac{1}{w_2-y_n} & \cdots & \frac{1}{w_{n+1}-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_{n+1}-y_1} \end{bmatrix} = \\ & = \det \begin{bmatrix} \frac{1}{z-w_1} & \frac{1}{z-w_2} & \cdots & \frac{1}{z-w_{n+1}} \\ \frac{z-y_n}{(z-w_1)(w_1-y_n)} & \frac{z-y_n}{(z-w_2)(w_2-y_n)} & \cdots & \frac{z-y_n}{(z-w_{n+1})(w_{n+1}-y_n)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{z-y_1}{(z-w_1)(w_1-y_1)} & \frac{z-y_1}{(z-w_2)(w_2-y_1)} & \cdots & \frac{z-y_1}{(z-w_{n+1})(w_{n+1}-y_1)} \end{bmatrix} = \\ & = \frac{\prod_{i=1}^n (z - y_i)}{\prod_{i=1}^{n+1} (z - w_i)} \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{w_1-y_n} & \frac{1}{w_2-y_n} & \cdots & \frac{1}{w_{n+1}-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1-y_1} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_{n+1}-y_1} \end{bmatrix}. \end{aligned}$$

Next, we subtract the first column from each of the other columns. This gives us

$$\begin{aligned} & \frac{\prod_{i=1}^n (z - y_i)}{\prod_{i=1}^{n+1} (z - w_i)} \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{w_1 - y_n} & \frac{w_1 - w_2}{(w_2 - y_n)(w_1 - y_n)} & \cdots & \frac{w_1 - w_{n+1}}{(w_{n+1} - y_1)(w_1 - y_n)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_1 - y_1} & \frac{w_1 - w_2}{(w_2 - y_1)(w_1 - y_1)} & \cdots & \frac{w_1 - w_{n+1}}{(w_1 - y_1)(w_{n+1} - y_1)} \end{bmatrix} = \\ & = \frac{\prod_{i=1}^n (z - y_i) \prod_{i=2}^{n+1} (w_1 - w_i)}{\prod_{i=1}^{n+1} (z - w_i) \prod_{i=1}^n (w_1 - y_i)} \det \begin{bmatrix} \frac{1}{w_2 - y_n} & \cdots & \frac{1}{w_{n+1} - y_n} \\ \vdots & \cdots & \vdots \\ \frac{1}{w_2 - y_1} & \cdots & \frac{1}{w_{n+1} - y_1} \end{bmatrix}. \end{aligned}$$

The determinant here is a Cauchy determinant equal to

$$\frac{\prod_{2 \leq i < j \leq n+1} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=2}^{n+1} \prod_{j=1}^n (w_i - y_j)}.$$

So our original determinant is equal to

$$\begin{aligned} & \frac{\prod_{j=n+2}^m \prod_{i < j} (w_i - w_j)}{\prod_{i=n+2}^m (z - w_i) \prod_{i=n+2}^m \prod_{j=1}^n (w_i - y_j)} \times \frac{\prod_{i=1}^n (z - y_i) \prod_{i=2}^{n+1} (w_1 - w_i)}{\prod_{i=1}^{n+1} (z - w_i) \prod_{i=1}^n (w_1 - y_i)} \times \frac{\prod_{2 \leq i < j \leq n+1} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=2}^{n+1} \prod_{j=1}^n (w_i - y_j)} = \\ & = \frac{\prod_{i=1}^n (z - y_i) \prod_{1 \leq i < j \leq m} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^m (z - w_i) \prod_{i=1}^m \prod_{j=1}^n (w_i - y_j)}. \end{aligned}$$

□

Lemma F.4. When $m \leq n$,

$$(-1)^m \det \begin{bmatrix} \frac{-1}{z-w_m} & \frac{1}{w_m-y_1} & \cdots & \frac{1}{w_m-y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{-1}{z-w_2} & \frac{1}{w_2-y_1} & \cdots & \frac{1}{w_2-y_n} \\ \frac{-1}{z-w_1} & \frac{1}{w_1-y_1} & \cdots & \frac{1}{w_1-y_n} \\ z^{n-m} & y_1^{n-m} & \cdots & y_n^{n-m} \\ z^{n-m-1} & y_1^{n-m-1} & \cdots & y_n^{n-m-1} \\ \vdots & \vdots & \cdots & \vdots \\ z & y_1 & \cdots & y_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \frac{\prod_{i=1}^n (z - y_i) \prod_{1 \leq i < j \leq m} (w_i - w_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^m (z - w_i) \prod_{i=1}^m \prod_{j=1}^n (w_i - y_j)}.$$

Proof. Subtract the first column from each of the others to obtain

$$(-1)^m \det \begin{bmatrix} \frac{-1}{z-w_m} & \frac{z-y_1}{(z-w_m)(w_m-y_1)} & \cdots & \frac{z-y_n}{(z-w_m)(w_m-y_n)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{-1}{z-w_2} & \frac{z-y_1}{(z-w_2)(w_2-y_1)} & \cdots & \frac{z-y_n}{(z-w_2)(w_2-y_n)} \\ \frac{-1}{z-w_1} & \frac{z-y_1}{(z-w_1)(w_1-y_1)} & \cdots & \frac{z-y_n}{(z-w_1)(w_1-y_n)} \\ z^{n-m} & y_1^{n-m} - z^{n-m} & \cdots & y_n^{n-m} - z^{n-m} \\ \vdots & \vdots & \cdots & \vdots \\ z & y_1 - z & \cdots & y_n - z \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Expanding along the last row and pulling out $\frac{\prod_{i=1}^n (z - y_i)}{\prod_{i=1}^m (z - w_i)}$ gives us

$$(-1)^{n+m} \frac{\prod_{i=1}^n (z - y_i)}{\prod_{i=1}^m (z - w_i)} \times$$

$$\begin{aligned}
& \det \begin{bmatrix} \frac{1}{w_m - y_1} & \frac{1}{w_m - y_2} & \cdots & \frac{1}{w_m - y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_2 - y_1} & \frac{1}{w_2 - y_2} & \cdots & \frac{1}{w_2 - y_n} \\ \frac{1}{w_1 - y_1} & \frac{1}{w_1 - y_2} & \cdots & \frac{1}{w_1 - y_n} \\ -\sum_{j=0}^{n-m-1} y_1^{n-m-1-j} z^j & -\sum_{j=0}^{n-m-1} y_2^{n-m-1-j} z^j & \cdots & -\sum_{j=0}^{n-m-1} y_n^{n-m-1-j} z^j \\ \vdots & \vdots & \cdots & \vdots \\ -y_1 - z & -y_2 - z & \cdots & -y_n - z \\ -1 & -1 & \cdots & -1 \end{bmatrix} = \\
& = \frac{\prod_{i=1}^n (z - y_i)}{\prod_{i=1}^m (z - w_i)} \det \begin{bmatrix} \frac{1}{w_m - y_1} & \frac{1}{w_m - y_2} & \cdots & \frac{1}{w_m - y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_2 - y_1} & \frac{1}{w_2 - y_2} & \cdots & \frac{1}{w_2 - y_n} \\ \frac{1}{w_1 - y_1} & \frac{1}{w_1 - y_2} & \cdots & \frac{1}{w_1 - y_n} \\ \sum_{j=0}^{n-m-1} y_1^{n-m-1-j} z^j & \sum_{j=0}^{n-m-1} y_2^{n-m-1-j} z^j & \cdots & \sum_{j=0}^{n-m-1} y_n^{n-m-1-j} z^j \\ \vdots & \vdots & \cdots & \vdots \\ y_1 + z & y_2 + z & \cdots & y_n + z \\ 1 & 1 & \cdots & 1 \end{bmatrix} =
\end{aligned}$$

Subtract z times the $(m+2)$ nd row from the $(m+1)$ st row, \cdots , and subtract z times the last row from the 2nd to last row. This gives us

$$\frac{\prod_{i=1}^n (z - y_i)}{\prod_{i=1}^m (z - w_i)} \det \begin{bmatrix} \frac{1}{w_m - y_1} & \frac{1}{w_m - y_2} & \cdots & \frac{1}{w_m - y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{w_2 - y_1} & \frac{1}{w_2 - y_2} & \cdots & \frac{1}{w_2 - y_n} \\ \frac{1}{w_1 - y_1} & \frac{1}{w_1 - y_2} & \cdots & \frac{1}{w_1 - y_n} \\ y_1^{n-m-1} & y_2^{n-m-1} & \cdots & y_n^{n-m-1} \\ \vdots & \vdots & \cdots & \vdots \\ y_1 & y_2 & \cdots & y_n \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We then apply Lemma G.1. to see that this is equal to

$$\frac{\prod_{i=1}^n (z - y_i) \prod_{1 \leq i < j \leq n} (y_i - y_j) \prod_{1 \leq i < j \leq m} (w_i - w_j)}{\prod_{i=1}^m (z - w_i) \prod_{i=1}^m \prod_{j=1}^n (w_i - y_j)}.$$



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